QUASIMODULAR FORMS, JACOBI-LIKE FORMS,
AND PSEUDODIFFERENTIAL OPERATORS

YOUNGJU CHOIE* AND MIN HO LEE†

ABSTRACT. We study various properties of quasimodular forms by using their connections with Jacobi-like forms and pseudodifferential operators. Such connections are made by identifying quasimodular forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ with certain polynomials over the ring of holomorphic functions of the Poincaré upper half plane that are Γ-invariant. We consider a surjective map from Jacobi-like forms to quasimodular forms and prove that it has a right inverse, which may be regarded as a lifting from quasimodular forms to Jacobi-like forms. We use such liftings to study Lie brackets and Rankin-Cohen brackets for quasimodular forms. We also discuss Hecke operators and construct Shimura isomorphisms and Shintani liftings for quasimodular forms.

1. Introduction

Quasimodular forms generalize classical modular forms and were introduced by Kaneko and Zagier in [13], where they identified quasimodular forms with generating functions counting maps of curves of genus $g > 1$ to curves of genus 1. More generally, they often appear as generating functions of counting functions in various problems such as Hurwitz enumeration problems, which include not only number theoretic problems but also those in certain areas of applied mathematics (see e.g. [3], [9],[10], [17], [21], [22]). Unlike modular forms, derivatives of quasimodular forms are also quasimodular forms. The goal of this paper is to study various properties of quasimodular forms by using their connections with Jacobi-like forms and pseudodifferential operators. Such connections are made by identifying quasimodular forms over a discrete subgroup Γ of $SL(2, \mathbb{R})$ with certain polynomials over the ring of holomorphic functions of the Poincaré upper half plane that are invariant under an action of Γ.

Jacobi-like forms for Γ are formal power series, whose coefficients are holomorphic functions on the Poincaré upper half plane $\mathcal{H}$, invariant under a certain right action of Γ (see [7], [25]). This invariance property is essentially one of the two conditions that must be satisfied by a Jacobi form (cf. [8]), and it determines relations among coefficients of the given Jacobi-like form. Such relations can be used to express each coefficient of a Jacobi-like form in terms of derivatives of some modular forms for Γ. These modular forms belong to a sequence of the form $\{f_w\}_{w \geq 0}$, where f_w is a modular form of weight $2w + \lambda$ for some $\lambda \in \mathbb{Z}$. In fact, sequences of modular forms for Γ of such type are in one-to-one correspondence with Jacobi-like forms for Γ of weight λ. Another interesting aspect of Jacobi-like forms is that

a Jacobi-like form for Γ determines a Γ-automorphic pseudodifferential operator, which is a formal Laurent series in ∂^{-1} having holomorphic functions on $\mathcal{H}$ as coefficients. There is a natural right action of $SL(2, \mathbb{R})$ on the space of pseudodifferential operators determined by linear factional transformations of $\mathcal{H}$, and Γ-automorphic pseudodifferential operators are pseudodifferential operators that are invariant under the action of $\Gamma \subset SL(2, \mathbb{R})$. We can consider an isomorphism between the space of formal power series and that of pseudodifferential operators, which provides a correspondence between Jacobi-like forms for Γ and Γ-automorphic pseudodifferential operators. Various topics related to mutual correspondences among Jacobi-like forms, pseudodifferential operators, and sequences of modular forms were investigated by Cohen, Manin, and Zagier (see [7] and [25]).

Given integers w and m with $m \geq 0$, a quasimodular form for Γ of weight w and depth at most m is a holomorphic function f on $\mathcal{H}$ associated to holomorphic functions $f_0, f_1, \ldots, f_m$ on $\mathcal{H}$ satisfying

$$\frac{1}{(cz+d)^w} f\left(\frac{az+b}{cz+d}\right) = f_0(z) + f_1(z)\left(\frac{c}{cz+d}\right) + \cdots + f_m(z)\left(\frac{c}{cz+d}\right)^m$$

for all $z \in \mathcal{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then the quasimodular form f determines the polynomial

$$F(z, X) = \sum_{r=0}^{m} f_r(z) X^r,$$

which we call a quasimodular polynomial. The condition for f to be a quasimodular form is equivalent to the condition that the polynomial $F(z, X)$ to be invariant under a certain right action of Γ on the space of polynomials of degree at most m with holomorphic coefficients. There is a natural surjective map from the space of Jacobi-like forms to that of quasimodular forms, and a similar map exists from pseudodifferential operators to quasimodular forms. One of the main results of this paper is the existence of the right inverse of such a surjective map, which may be regarded as a lifting from quasimodular forms to Jacobi-like forms or to pseudodifferential operators.

The right action of $SL(2, \mathbb{R})$ on the space of polynomials over holomorphic functions on $\mathcal{H}$ mentioned above can be used to define Hecke operators on quasimodular polynomials, which are compatible with the usual Hecke operators on modular forms. On the other hand, the action of $SL(2, \mathbb{R})$ on formal power series described earlier determines Hecke operators on Jacobi-like forms. We show that the surjective maps from Jacobi-like forms to quasimodular polynomials are Hecke equivariant. We also study various applications of the liftings from quasimodular forms to Jacobi-like forms. In particular, we introduce a Lie algebra structure on the space of quasimodular polynomials and construct Rankin-Cohen brackets for quasimodular forms.

As another application of liftings of quasimodular forms to Jacobi-like forms, we study the Shimura correspondence for quasimodular forms. Given an odd integer k, the usual Shimura correspondence associates to each modular form of weight $\lambda = (k-1)/2$ a modular form of weight $2\lambda - 1$. On the other hand, there is also a map carrying modular forms in the opposite direction known as the Shintani lifting. We introduce the notion of quasimodular forms and Jacobi-like forms of half integral weight and construct a quasimodular analog of Shimura isomorphisms and Shintani liftings.

This paper is organized as follows. In Section 2 we study formal power series and polynomials over the ring of holomorphic functions on $\mathcal{H}$. We consider linear maps between such

power series and polynomials and introduce actions of $SL(2, \mathbb{R})$ that are compatible with respect to these maps. In Section 3 we describe quasimodular forms for Γ and some of their properties. We introduce quasimodular polynomials, which can be identified with quasimodular forms. We also consider Γ-invariant formal power series known as Jacobi-like forms and discuss their connections with quasimodular polynomials. Section 4 is concerned with Hecke operators on quasimodular polynomials that are compatible with those on modular and Jacobi-like forms. In Section 5 we construct a lifting of a modular form to a quasimodular form whose leading coefficient coincides with the given modular form. The existence of liftings of quasimodular forms to Jacobi-like forms is one of our main findings and is proved in Section 6. The noncommutative multiplication operation on the space of pseudodifferential operators given by the Leibniz rule determines a natural Lie algebra structure on the same space. In Section 7 we use the correspondence between pseudodifferential operators and Jacobi-like forms as well as the liftings discussed in Section 6 to construct Lie brackets on the space of quasimodular polynomials. As another application of the liftings, we study Rankin-Cohen brackets for quasimodular forms in Section 8. We extend the notion of quasimodular and Jacobi-like forms to the case of half integral weight in Section 9, and investigate the quasimodular analog of correspondences of Shimura and Shintani in Sections 10 and 11.

2. Formal power series and polynomials

In this section we study formal power series and polynomials whose coefficients are holomorphic functions on the Poincaré upper half plane. We consider linear maps between such power series and polynomials and introduce actions of $SL(2, \mathbb{R})$ that are compatible with respect to these maps.

The group $SL(2, \mathbb{R})$ acts on the Poincaré upper half plane $\mathcal{H}$ as usual by linear fractional transformations given by

$$\gamma z = \frac{az + b}{cz + d}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{R})$. For the same z and γ, we set

$$(2.1) \qquad \mathfrak{J}(\gamma, z) = cz + d, \quad \mathfrak{K}(\gamma, z) = \mathfrak{J}(\gamma, z)^{-1}\frac{d}{dz}(\mathfrak{J}(\gamma, z)) = \frac{c}{cz + d},$$

so that we obtain the maps $\mathfrak{J}, \mathfrak{K} : SL(2, \mathbb{R}) \times \mathcal{H} \to \mathbb{C}$. As is well-known, the map $\mathfrak{J}$ is an automorphy factor satisfying the cocycle condition

$$(2.2) \qquad \mathfrak{J}(\gamma\gamma', z) = \mathfrak{J}(\gamma, \gamma'z)\mathfrak{J}(\gamma', z)$$

for all $\gamma, \gamma' \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$. On the other hand, it can be shown that the other map $\mathfrak{K}$ satisfies

$$(2.3) \qquad \mathfrak{K}(\gamma\gamma', z) = \mathfrak{K}(\gamma', z) + \mathfrak{J}(\gamma', z)^{-2}\mathfrak{K}(\gamma, \gamma'z).$$

Let $\mathcal{F}$ denote the ring of holomorphic functions on $\mathcal{H}$, and let $\mathcal{F}[[X]]$ be the complex algebra of formal power series in X with coefficients in $\mathcal{F}$. Given elements $f \in \mathcal{F}$, $\Phi(z, X) \in \mathcal{F}[[X]]$, $\lambda \in \mathbb{Z}$, and $\gamma \in SL(2, \mathbb{R})$, we set

$$(2.4) \qquad (f \mid_\lambda \gamma)(z) = \mathfrak{J}(\gamma, z)^{-\lambda}f(\gamma z)$$

$$(2.5) \qquad (\Phi \mid_\lambda^J \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda}e^{-\mathfrak{K}(\gamma, z)X}\Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)$$

for all $z \in \mathcal{H}$. If γ' is another element of $SL(2, \mathbb{R})$, then from (2.2) and (2.3) it can be shown that
$$f \mid_\lambda (\gamma\gamma') = (f \mid_\lambda \gamma) \mid_\lambda \gamma', \quad \Phi \mid_\lambda^J (\gamma\gamma') = (\Phi \mid_\lambda^J \gamma) \mid_\lambda^J \gamma';$$
hence the operations $\mid_\lambda$ and $\mid_\lambda^J$ determine right actions of $SL(2, \mathbb{R})$ on $\mathcal{F}$ and $\mathcal{F}[[X]]$, respectively. If δ is a nonnegative integer, we set
$$\mathcal{F}[[X]]_\delta = X^\delta \mathcal{F}[[X]],$$
so that an element $\Phi(z, X) \in \mathcal{F}[[X]]_\delta$ can be written in the form

(2.6)
$$\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta}$$

with $\phi_k \in \mathcal{F}$ for each $k \geq 0$.

Given a nonnegative integer m, we now consider the complex algebra $\mathcal{F}_m[X]$ of polynomials in X over $\mathcal{F}$ of degree at most m. If $\gamma \in SL(2, \mathbb{R})$, $\lambda \in \mathbb{Z}$ and $F(z, X) \in \mathcal{F}_m[X]$, we set

(2.7)
$$(F \|_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} F(\gamma z, \mathfrak{J}(\gamma, z)^2 (X - \mathfrak{K}(\gamma, z)))$$

for all $z \in \mathcal{H}$.

Lemma 2.1. *If $F(z, X) \in \mathcal{F}_m[X]$ and $\lambda \in \mathbb{Z}$, then we have*
$$((F \|_\lambda \gamma) \|_\lambda \gamma')(z, X) = (F \|_\lambda (\gamma\gamma'))(z, X)$$
for all $\gamma, \gamma' \in SL(2, \mathbb{R})$.

Proof. Given $\gamma, \gamma' \in SL(2, \mathbb{R})$, $\lambda \in \mathbb{Z}$ and $F(z, X) \in \mathcal{F}_m[X]$, using (2.7), we have

(2.8)
$$((F \|_\lambda \gamma) \|_\lambda \gamma')(z, X) = \mathfrak{J}(\gamma', z)^{-\lambda} (F \|_\lambda \gamma)(\gamma' z, \mathfrak{J}(\gamma', z)^2 (X - \mathfrak{K}(\gamma', z)))$$
$$= \mathfrak{J}(\gamma', z)^{-\lambda} \mathfrak{J}(\gamma, \gamma' z)^{-\lambda} F(\gamma\gamma' z, X'),$$

where
$$X' = \mathfrak{J}(\gamma, \gamma' z)^2 \mathfrak{J}(\gamma', z)^2 (X - \mathfrak{K}(\gamma', z)) - \mathfrak{J}(\gamma, \gamma' z)^2 \mathfrak{K}(\gamma, \gamma' z).$$
Thus using (2.2) and (2.3), we see that
$$X' = \mathfrak{J}(\gamma\gamma', z)^2 (X - \mathfrak{K}(\gamma', z)) - \mathfrak{J}(\gamma, \gamma' z)^2 \mathfrak{J}(\gamma', z)^2 (\mathfrak{K}(\gamma\gamma', z) - \mathfrak{K}(\gamma', z))$$
$$= \mathfrak{J}(\gamma\gamma', z)^2 (X - \mathfrak{K}(\gamma\gamma', z)).$$
From this and (2.8), we obtain
$$((F \|_\lambda \gamma) \|_\lambda \gamma')(z, X) = \mathfrak{J}(\gamma\gamma', z)^{-\lambda} F(\gamma\gamma' z, \mathfrak{J}(\gamma\gamma', z)^2 (X - \mathfrak{K}(\gamma\gamma', z))$$
$$= (F \|_\lambda (\gamma\gamma'))(z, X);$$
hence the lemma follows. $\square$

If $\Phi(z, X) \in \mathcal{F}[[X]]_\delta$ is as in (2.6) and if m is a nonnegative integer, we set

(2.9)
$$(\Pi_m^\delta \Phi)(z, X) = \sum_{r=0}^{m} \frac{1}{r!} \phi_{m-r}(z) X^r,$$

which determines the surjective complex linear map
$$\Pi_m^\delta : \mathcal{F}[[X]]_\delta \to \mathcal{F}_m[X].$$
In particular, for $m = 0$ we obtain the map
$$\Pi_0^\delta : \mathcal{F}[[X]]_\delta \to \mathcal{F}$$

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given by

$$(\Pi_0^\delta \Phi)(z) = \phi_0(z).$$

We also define the linear maps

(2.10)
$$\mathfrak{S}_\ell : \mathcal{F}_m[X] \to \mathcal{F}$$

for $0 \le \ell \le m$ by setting

(2.11)
$$(\mathfrak{S}_\ell F)(z) = f_\ell(z)$$

for $\Phi(z, X) \in \mathcal{F}[[X]]_\delta$ as above and $F(z, X) = \sum_{k=0}^m f_k(z) X^k \in \mathcal{F}_m[X]$.

Lemma 2.2. *Given $m, \delta \ge 0$, the diagram*

(2.12)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}[[X]]_{\delta+1} & \stackrel{\widehat{\iota}}{\longrightarrow} & \mathcal{F}[[X]]_\delta & \stackrel{\widehat{\mathfrak{S}}^\delta}{\longrightarrow} & \mathcal{F} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \Pi_{m-1}^{\delta+1}} & & \downarrow{\scriptstyle \Pi_m^\delta} & & \downarrow{\scriptstyle \mu_m} & & \\
0 & \longrightarrow & \mathcal{F}_{m-1}[X] & \stackrel{\iota}{\longrightarrow} & \mathcal{F}_m[X] & \stackrel{\mathfrak{S}_m}{\longrightarrow} & \mathcal{F} & \longrightarrow & 0
\end{array}
$$

commutes, where $\widehat{\iota}$ and ι are inclusion maps and μ_m is multiplication by $(1/m!)$. Furthermore, the two rows in the diagram are short exact sequences.

Proof. Let $\Phi(z, X) \in \mathcal{F}[[X]]_\delta$ and $\Psi(z, X) \in \mathcal{F}[[X]]_{\delta+1}$ be given by

$$\Phi(z, X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta}, \quad \Psi(z, X) = \sum_{k=0}^\infty \psi_k(z) X^{k+\delta+1}.$$

Then from (2.9) and (2.11) we see that

$$((\mathfrak{S}_m \circ \Pi_m^\delta) \Phi)(z) = \phi_0(z)/m! = (\mu_m \phi_0)(z).$$

Since $\phi_0 = \widehat{\mathfrak{S}}^\delta \Phi$ by (2.11), we obtain

$$\mathfrak{S}_m \circ \Pi_m^\delta = \mu_m \circ \widehat{\mathfrak{S}}^\delta.$$

On the other hand, we have

$$(\widehat{\iota}\Psi)(z, X) = \sum_{k=0}^\infty \widehat{\psi}_k(z) X^{k+\delta}$$

with $\widehat{\psi}_0 = 0$ and $\widehat{\psi}_k = \psi_{k-1}$ for $1 \le k \le m$. Hence we see that

$$((\Pi_m^\delta \circ \widehat{\iota})\Psi)(z, X) = \sum_{r=0}^m \frac{1}{r!} \widehat{\psi}_{m-r}(z) X^r = \sum_{r=0}^{m-1} \frac{1}{r!} \psi_{m-1-r}(z) X^r$$
$$= (\Pi_{m-1}^{\delta+1} \Psi)(z, X) = ((\iota \circ \Pi_{m-1}^{\delta+1})\Psi)(z, X),$$

which shows that $\Pi_m^\delta \circ \widehat{\iota} = \iota \circ \Pi_{m-1}^{\delta+1}$. The two rows in the diagram are clearly short exact sequences. $\square$

The next proposition shows that the surjective map $\Pi_m^\delta : \mathcal{F}[[X]]_\delta \to \mathcal{F}_m[X]$ is equivariant with respect to the right actions of $SL(2, \mathbb{R})$ of the type described earlier.

Proposition 2.3. *If m, δ and λ are integers with $m, \delta \ge 0$, we have*

(2.13)
$$\Pi_m^\delta(\Phi \mid_\lambda^J \gamma) = \Pi_m^\delta(\Phi) \parallel_{\lambda+2m+2\delta} \gamma$$

for all $\Phi(z, X) \in \mathcal{F}[[X]]_\delta$ and $\gamma \in SL(2, \mathbb{R})$.

Proof. Let $\Phi(z, X) \in \mathcal{F}[[X]]_\delta$ be as in (2.6), and let $\gamma \in SL(2, \mathbb{R})$. Then, using (2.5), we have

$$(\Phi \mid_\lambda^J \gamma)(z, X) = \sum_{k=0}^\infty \mathfrak{J}(\gamma, z)^{-\lambda} \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!} \mathfrak{K}(\gamma, z)^\ell X^\ell \phi_k(\gamma z) \mathfrak{J}(\gamma, z)^{-2k-2\delta} X^{k+\delta}$$

$$= \sum_{k=0}^\infty \sum_{\ell=0}^\infty \frac{(-1)^\ell}{\ell!} \mathfrak{J}(\gamma, z)^{-2k-2\delta-\lambda} \mathfrak{K}(\gamma, z)^\ell \phi_k(\gamma z) X^{k+\ell+\delta}$$

$$= \sum_{n=0}^\infty \widetilde{\phi}_n(z, X) X^{n+\delta}$$

with

$$\widetilde{\phi}_n(z, X) = \sum_{\ell=0}^n \frac{(-1)^\ell}{\ell!} \mathfrak{J}(\gamma, z)^{-2n+2\ell-2\delta-\lambda} \mathfrak{K}(\gamma, z)^\ell \phi_{n-\ell}(\gamma z).$$

From this and (2.9) we see that

$$(2.14) \qquad (\Pi_m^\delta(\Phi \mid_\lambda^J \gamma))(z, X) = \sum_{r=0}^m \frac{1}{r!} \widetilde{\phi}_{m-r}(z) X^r$$

$$= \sum_{r=0}^m \sum_{\ell=0}^{m-r} \frac{(-1)^\ell}{r!\ell!} \mathfrak{J}(\gamma, z)^{2\ell+2r-2m-2\delta-\lambda} \mathfrak{K}(\gamma, z)^\ell \phi_{m-\ell-r}(\gamma z) X^r.$$

On the other hand, using (2.7) and (2.9), we have

$$((\Pi_m^\delta \Phi) \|_{\lambda+2m+2\delta} \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda-2m-2\delta}(\Pi_m^\delta \Phi)(\gamma z, \mathfrak{J}(\gamma, z)^2(X - \mathfrak{K}(\gamma, z)))$$

$$= \mathfrak{J}(\gamma, z)^{-\lambda-2m-2\delta} \sum_{j=0}^m \frac{1}{j!} \phi_{m-j}(\gamma z) \mathfrak{J}(\gamma, z)^{2j}(X - \mathfrak{K}(\gamma, z))^j$$

$$= \sum_{j=0}^m \sum_{r=0}^j \frac{1}{j!}\binom{j}{r} \phi_{m-j}(\gamma z) \mathfrak{J}(\gamma, z)^{2j-\lambda-2m-2\delta}$$

$$\times (-1)^{j-r} \mathfrak{K}(\gamma, z)^{j-r} X^r$$

$$= \sum_{r=0}^m \sum_{j=r}^m \frac{(-1)^{j-r}}{r!(j-r)!} \mathfrak{J}(\gamma, z)^{2j-\lambda-2m-2\delta} \mathfrak{K}(\gamma, z)^{j-r} \phi_{m-j}(\gamma z) X^r.$$

Changing the index for the second summation in the previous line from j to $\ell = j - r$ and comparing this with (2.14), we obtain the relation (2.13). $\square$

3. Quasimodular forms

Jacobi-like forms for Γ are formal power series, whose coefficients are holomorphic functions on the Poincaré upper half plane $\mathcal{H}$, invariant under a certain right action of Γ (see [7], [25]).

In this section we describe quasimodular forms for a discrete subgroup Γ of $SL(2, \mathbb{R})$ introduced by Kaneko and Zagier (see [13]) and study some of their properties. We introduce quasimodular polynomials, which are polynomials with holomorphic coefficients invariant under a right action of Γ and can be identified with quasimodular forms. We also consider Γ-invariant formal power series known as Jacobi-like forms (cf. [7], [25]) and discuss their connections with quasimodular polynomials.

Let $\mathcal{F}$ be the ring of holomorphic functions on $\mathcal{H}$ as in Section 2, and let Γ be a discrete subgroup of $SL(2, \mathbb{R})$. We also fix a nonnegative integer m.

Definition 3.1. Let ξ be an integer, and let $|_\xi$ be the operation in (2.4).

(i) An element $f \in \mathcal{F}$ is a *modular form for Γ of weight ξ* if it satisfies

$$f \mid_\xi \gamma = f$$

for all $\gamma \in \Gamma$. We denote by $M_\xi(\Gamma)$ the space of modular forms for Γ of weight ξ.

(ii) An element $f \in \mathcal{F}$ is a *quasimodular form for Γ of weight ξ and depth at most m* if there are functions $f_0, \ldots, f_m \in \mathcal{F}$ such that

$$(3.1) \qquad (f \mid_\xi \gamma)(z) = \sum_{r=0}^{m} f_r(z) \mathfrak{K}(\gamma, z)^r$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\mathfrak{K}(\gamma, z)$ is as in (2.1). We denote by $QM_\xi^m(\Gamma)$ the space of quasimodular forms for Γ of weight ξ and depth at most m.

Remark 3.2. (i) Although the usual definition of modular forms or quasimodular forms includes a cusp condition, we have suppressed such conditions in Definition 3.1.

(ii) If we set $\gamma \in \Gamma$ in (3.1) to be the identity matrix, then $\mathfrak{K}(\gamma, z) = 0$; hence it follows that

$$f = f_0.$$

If $m = 0$, from (3.1) we obtain

$$f \mid_\xi \gamma = f_0 = f,$$

and therefore we have

$$QM_\xi^0(\Gamma) = M_\xi(\Gamma).$$

(iii) For fixed $z \in \mathcal{H}$, by considering the right hand side of (3.1) as a polynomial in $\mathfrak{K}(\gamma, z)$ and using the fact that it is valid for all γ in the infinite set Γ, we see that f determines the coefficients $f_0, \ldots, f_m$ uniquely.

Let $f \in \mathcal{F}$ be a quasimodular form belonging to $QM_\xi^m(\Gamma)$ satisfying (3.1). Then we define the corresponding polynomial $(\mathcal{Q}_\xi^m f)(z, X) \in \mathcal{F}_m[X]$ by

$$(3.2) \qquad (\mathcal{Q}_\xi^m f)(z, X) = \sum_{r=0}^{m} f_r(z) X^r$$

for $z \in \mathcal{H}$. We note that $\mathcal{Q}_\xi^m f$ is well-defined due to Remark 3.2(iii). Thus we obtain the linear map

$$\mathcal{Q}_\xi^m : QM_\xi^m(\Gamma) \to \mathcal{F}_m[X]$$

for each $\xi \in \mathbb{Z}$.

Definition 3.3. A *quasimodular polynomial for Γ of weight ξ and degree at most m* is an element of $\mathcal{F}_m[X]$ that is invariant with respect to the right Γ-action in (2.7). We denote by

$$QP_\xi^m(\Gamma) = \{F(z, X) \in \mathcal{F}_m[X] \mid F \parallel_\xi \gamma = F \text{ for all } \gamma \in \Gamma\}$$

the space of all quasimodular polynomials for Γ of weight ξ and degree at most m, where $\parallel_\xi$ is as in (2.7).

Lemma 3.4. *Given $F(z, X) = \sum_{r=0}^{m} f_r(z) X^r \in \mathcal{F}_m[X]$ and $\gamma \in SL(2, \mathbb{R})$, we have*

(3.3)
$$\mathfrak{S}_r(F \|_\xi \gamma)(z) = \sum_{\ell=r}^{m} (-1)^{\ell-r} \binom{\ell}{r} f_\ell(\gamma z) \mathfrak{J}(\gamma, z)^{2\ell-\xi} \mathfrak{K}(\gamma, z)^{\ell-r},$$

(3.4)
$$(\mathfrak{S}_r(F \|_\xi \gamma^{-1}) \mid_{\xi-2r} \gamma)(z) = \sum_{\ell=r}^{m} \binom{\ell}{r} f_\ell(z) \mathfrak{K}(\gamma, z)^{\ell-r}$$

for $0 \le r \le m$.

Proof. Using (2.7), we have

$$(F \|_\xi \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\xi} \sum_{\ell=0}^{m} f_\ell(\gamma z) \mathfrak{J}(\gamma, z)^{2\ell} (X - \mathfrak{K}(\gamma, z))^\ell$$

$$= \sum_{\ell=0}^{m} \sum_{r=0}^{\ell} \binom{\ell}{r} f_\ell(\gamma z) \mathfrak{J}(\gamma, z)^{2\ell-\xi} (-1)^{\ell-r} \mathfrak{K}(\gamma, z)^{\ell-r} X^r$$

$$= \sum_{r=0}^{m} \sum_{\ell=r}^{m} (-1)^{\ell-r} \binom{\ell}{r} f_\ell(\gamma z) \mathfrak{J}(\gamma, z)^{2\ell-\xi} \mathfrak{K}(\gamma, z)^{\ell-r} X^r,$$

which verifies (3.3). Replacing γ by γ^{-1} and z by γz in (3.3), we have

$$\mathfrak{S}_r(F \|_\xi \gamma^{-1})(\gamma z) = \mathfrak{J}(\gamma^{-1}, \gamma z)^{-\xi} \sum_{\ell=r}^{m} (-1)^{\ell-r} \binom{\ell}{r} f_\ell(z) \mathfrak{J}(\gamma^{-1}, \gamma z)^{2\ell} \mathfrak{K}(\gamma^{-1}, \gamma z)^{\ell-r}.$$

However, from (2.2) and (2.3) and the fact that $\mathfrak{J}(1, z) = 1$ and $\mathfrak{K}(1, z) = 0$, we see that

(3.5)
$$\mathfrak{J}(\gamma^{-1}, \gamma z) = \mathfrak{J}(\gamma, z)^{-1}, \quad \mathfrak{K}(\gamma^{-1}, \gamma z) = -\mathfrak{J}(\gamma, z)^2 \mathfrak{K}(\gamma, z),$$

and therefore it follows that

$$\mathfrak{S}_r(F \|_\xi \gamma^{-1})(\gamma z) = \mathfrak{J}(\gamma, z)^\xi \sum_{\ell=r}^{m} (-1)^{\ell-r} \binom{\ell}{r} f_\ell(z) \mathfrak{J}(\gamma, z)^{-2\ell} (-1)^{\ell-r} \mathfrak{J}(\gamma, z)^{2\ell-2r} \mathfrak{K}(\gamma, z)^{\ell-r}$$

$$= \mathfrak{J}(\gamma, z)^{\xi-2r} \sum_{\ell=r}^{m} \binom{\ell}{r} f_\ell(z) \mathfrak{K}(\gamma, z)^{\ell-r};$$

hence we obtain (3.4). $\qquad\square$

By setting $r = 0$ in (3.3) we obtain

(3.6)
$$\mathfrak{S}_0(F \|_\xi \gamma)(z) = \sum_{\ell=0}^{m} (-1)^\ell \mathfrak{J}(\gamma, z)^{2\ell-\xi} \mathfrak{K}(\gamma, z)^\ell f_\ell(\gamma z),$$

$$\mathfrak{S}_0(F \|_\xi \gamma^{-1})(\gamma z) = \mathfrak{J}(\gamma, z)^\xi \sum_{r=0}^{m} f_r(z) \mathfrak{K}(\gamma, z)^r,$$

where we used (3.5). In particular, if $F(z, X) \in QP_\xi^m(\Gamma)$, we have

$$(\mathfrak{S}_0 F)(z) = \sum_{\ell=0}^{m} (-1)^\ell \mathfrak{J}(\gamma, z)^{2\ell-\xi} \mathfrak{K}(\gamma, z)^\ell f_\ell(\gamma z)$$

for all $\gamma \in \Gamma$.

Corollary 3.5. *Let $F(z, X) = \sum_{r=0}^{m} f_r(z) X^r \in \mathcal{F}_m[X]$. Then $F(z, X)$ is a quasimodular polynomial belonging to $QP_\xi^m(\Gamma)$ if and only if for each $r \in \{0, 1, \dots, m\}$ the coefficient f_r satisfies*

$$(3.7) \qquad (f_r \mid_{\xi-2r} \gamma)(z) = \sum_{\ell=r}^{m} \binom{\ell}{r} f_\ell(z) \mathfrak{K}(\gamma, z)^{\ell-r} = \sum_{\ell=0}^{m-r} \binom{\ell+r}{r} f_{\ell+r}(z) \mathfrak{K}(\gamma, z)^\ell$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. In particular, f_r is a quasimodular form belonging to $QM_{\xi-2r}^{m-r}(\Gamma)$.

Proof. A polynomial $F(z, X) \in \mathcal{F}_m[X]$ belongs to $QP_\xi^m(\Gamma)$ if an only if

$$F(z, X) = (F \parallel_\xi \gamma)(z, X)$$

for all $\gamma \in \Gamma$. Hence (3.7) follows from this and (3.4). $\qquad\square$

If $0 \le \ell \le m$, we obtain the complex linear map

$$\mathfrak{S}_\ell : QP_\xi^m(\Gamma) \to \mathcal{F}$$

by restricting $\mathfrak{S}_\ell$ in (2.10) to $QP_\xi^m(\Gamma)$. Then $\mathfrak{S}_r F$ belongs to $QM_{\xi-2r}^{m-r}(\Gamma)$ for $0 \le r \le m$, and (3.7) can be written in the form

$$(\mathfrak{S}_r F \mid_{\xi-2r} \gamma)(z) = \sum_{\ell=0}^{m-r} \binom{\ell+r}{r} (\mathfrak{S}_{\ell+r} F)(z) \mathfrak{K}(\gamma, z)^\ell$$

for $F(z, X) \in QP_\xi^m(\Gamma)$. In particular, we obtain

$$\mathfrak{S}_m F \mid_{\xi-2m} \gamma = \mathfrak{S}_m F$$

for all $\gamma \in \Gamma$. Thus it follows that

$$(3.8) \qquad\qquad\qquad \mathfrak{S}_m F \in M_{\xi-2m}(\Gamma)$$

if $F(z, X) \in QP_\xi^m(\Gamma)$. On the other hand, we also have

$$(\mathcal{Q}_{\xi-2r}^{m-r}(\mathfrak{S}_r F))(z, X) = \sum_{\ell=0}^{m-r} \binom{\ell+r}{r} (\mathfrak{S}_{\ell+r} F)(z) X^\ell \in QP_{\xi-2r}^{m-r}(\Gamma),$$

where $\mathcal{Q}_{\xi-2r}^{m-r}$ is as in (3.2). Thus, in particular, we see that the map $\mathcal{Q}_\xi^m$ given by (3.2) determines the complex linear map

$$(3.9) \qquad\qquad\qquad \mathcal{Q}_\xi^m : QM_\xi^m(\Gamma) \to QP_\xi^m(\Gamma),$$

so that $\mathcal{Q}_\xi^m$ carries quasimodular forms to quasimodular polynomials. In fact, the the next proposition shows that it is an isomorphism.

Proposition 3.6. *The restriction of the map $\mathfrak{S}_\ell$ in (2.10) with $\ell = 0$ to $QP_\xi^m(\Gamma)$ determines an isomorphism*

$$(3.10) \qquad\qquad\qquad \mathfrak{S}_0 : QP_\xi^m(\Gamma) \to QM_\xi^m(\Gamma)$$

whose inverse is the map $\mathcal{Q}_\xi^m$ in (3.9).

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Proof. If $F(z, X) = \sum_{r=0}^{m} f_r(z)X^r$ is an element of $QP_\xi^m(\Gamma)$, then we have

$$(\mathfrak{S}_0 F)(z) = f_0(z)$$

for $z \in \mathcal{H}$. However, from Corollary 3.5 we see that

$$(f_0 \mid_\xi \gamma)(z) = \sum_{\ell=0}^{m} f_\ell(z)\mathfrak{K}(\gamma, z)^\ell$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. Thus we have

$$((\mathcal{Q}_\xi^m \circ \mathfrak{S}_0)F)(z, X) = (\mathcal{Q}_\xi^m f_0)(z, X) = \sum_{j=0}^{m} f_j(z)X^j = F(z, X).$$

We now consider an element $h \in QM_\xi^m(\Gamma)$ satisfying

$$(h \mid_\xi \gamma)(z) = \sum_{\ell=0}^{m} h_\ell(z)\mathfrak{K}(\gamma, z)^\ell$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, so that

$$(\mathcal{Q}_\xi^m h)(z, X) = \sum_{\ell=0}^{m} h_\ell(z)X^\ell.$$

Then we see that

$$(\mathfrak{S}_0(\mathcal{Q}_\xi^m h))(z) = h_0(z) = h(z)$$

for all $z \in \mathcal{H}$, and therefore the proposition follows. $\qquad\square$

Corollary 3.7. *Given* $F(z, X) = \sum_{r=0}^{m} f_r(z)X^r \in QP_{\lambda+2m+2\delta}^m(\Gamma)$, *we set*

(3.11) $$\phi_i = (m - i)! f_{m-i}$$

for $0 \leq i \leq m$. *Then for each* $k \in \{0, 1, \ldots, m\}$ *we have*

(3.12) $$(\phi_k \mid_{2k+2\delta+\lambda} \gamma)(z) = \sum_{r=0}^{k} \frac{1}{r!}\mathfrak{K}(\gamma, z)^r \phi_{k-r}(z)$$

for all $z \in \mathcal{H}$ *and* $\gamma \in \Gamma$.

Proof. If $0 \leq k \leq m$, using (3.7) and (3.11), we have

$$(\phi_k \mid_{2k+2\delta+\lambda} \gamma)(z) = (m - k)!(f_{m-k} \mid_{2k+2\delta+\lambda} \gamma)(z)$$

$$= (m - k)! \sum_{\ell=m-k}^{m} \binom{\ell}{m - k} f_\ell(z)\mathfrak{K}(\gamma, z)^{\ell-m+k}$$

$$= \sum_{\ell=m-k}^{m} \frac{\phi_{m-\ell}(z)}{(\ell - m + k)!}\mathfrak{K}(\gamma, z)^{\ell-m+k}$$

for all $z \in \mathcal{H}$. Hence (3.12) can be obtained by changing the index from ℓ to $r = \ell - m + k$ in the previous sum. $\qquad\square$

Remark 3.8. Let $f \in QM_\xi^m(\Gamma)$ be a quasimodular form satisfying (3.1). In the proof of Theorem 1 in [1], it was indicated that the corresponding polynomial $F(z, X) = (\mathcal{Q}_\xi^m f)(z, X) \in \mathcal{F}_m[X]$ satisfies

(3.13) $$\mathfrak{J}(\gamma, z)^{-\xi} F(\gamma z, X) = F(z, \mathfrak{J}(\gamma, z)^{-2} X + \mathfrak{K}(\gamma, z))$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. It can be shown, however, that (3.13) is equivalent to the condition that

$$F \,\|_\xi\, \gamma = F$$

for all $\gamma \in \Gamma$. Indeed, if $F(z, X) \in \mathcal{F}_m[X]$ satisfies (3.13), by replacing γ and z by γ^{-1} and γz, respectively, we have

(3.14) $$\mathfrak{J}(\gamma^{-1}, \gamma z)^{-\xi} F(z, X) = F(\gamma z, \mathfrak{J}(\gamma^{-1}, \gamma z)^{-2} X + \mathfrak{K}(\gamma^{-1}, \gamma z)).$$

From (3.5) and (3.14) we obtain

$$F(z, X) = \mathfrak{J}(\gamma, z)^{-\xi} F(\gamma z, \mathfrak{J}(\gamma, z)^2 X - \mathfrak{J}(\gamma, z)^2 \mathfrak{K}(\gamma, z)) = (F \,\|_\xi\, \gamma)(z, X),$$

where we used (2.7).

We now consider Jacobi-like forms studied by Cohen, Manin and Zagier in [7] and [25]. It turns out that they are closely linked to quasimodular polynomials and therefore to quasimodular forms.

Definition 3.9. Given an integer λ, a formal power series $\Phi(z, X)$ belonging to $\mathcal{F}[[X]]$ is a *Jacobi-like form for Γ of weight λ* if it satisfies

$$(\Phi \,|_\lambda^J\, \gamma)(z, X) = \Phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $|_\lambda^J$ is as in (2.5).

We denote by $\mathcal{J}_\lambda(\Gamma)$ the space of all Jacobi-like forms for Γ of weight λ, and set

$$\mathcal{J}_\lambda(\Gamma)_\delta = \mathcal{J}_\lambda(\Gamma) \cap \mathcal{F}[[X]]_\delta$$

for each nonnegative integer δ.

Proposition 3.10. *The map Π_m^δ in (2.9) induces the complex linear map*

(3.15) $$\Pi_m^\delta : \mathcal{J}_\lambda(\Gamma)_\delta \to QP_{\lambda+2m+2\delta}^m(\Gamma)$$

for each $\delta \geq 0$ and $\lambda \in \mathbb{Z}$.

Proof. Given $\delta, \lambda \in \mathbb{Z}$ with $\delta \geq 0$, we need to show that

$$\Pi_m^\delta(\mathcal{J}_\lambda(\Gamma)_\delta) \subset QP_{\lambda+2m+2\delta}^m(\Gamma).$$

However, this follows immediately from Proposition 2.3, Definition 3.3 and Definition 3.9. $\quad\square$

4. Hecke operators

Hecke operators acting on the space of modular forms play an important role in number theory, and similar operators can also be defined on the space of Jacobi-like forms. In this section we introduce Hecke operators on quasimodular polynomials that are compatible with those on modular and Jacobi-like forms.

Let $GL^+(2, \mathbb{R})$ be the group of 2×2 real matrices of positive determinant, which acts on the Poincaré upper half plane $\mathcal{H}$ by linear fractional transformations. We extend the functions $\mathfrak{J}$ and $\mathfrak{K}$ given by (2.1) to the maps $\mathfrak{J}, \mathfrak{K} : GL^+(2, \mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ by setting

$$(4.1) \qquad \mathfrak{J}(\alpha, z) = cz + d, \quad \mathfrak{K}(\alpha, z) = \mathfrak{J}(\alpha, z)^{-1} \frac{d}{dz}(\mathfrak{J}(\alpha, z))$$

for $z \in \mathcal{H}$ and $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in GL^+(2, \mathbb{R})$. Then it can be shown that $\mathfrak{J}$ satisfies the cocycle condition

$$\mathfrak{J}(\alpha\alpha', z) = \mathfrak{J}(\alpha, \alpha'z)\mathfrak{J}(\alpha', z)$$

for all $z \in \mathcal{H}$ and $\alpha, \alpha' \in GL^+(2, \mathbb{R})$ as in (2.2). On the other hand, the relation (2.3) should be modified as

$$\mathfrak{K}(\alpha\alpha', z) = \mathfrak{K}(\alpha', z) + (\det \alpha')\mathfrak{J}(\alpha', z)^{-2}\mathfrak{K}(\alpha, \alpha'z).$$

Two subgroups Γ_1 and Γ_2 of $GL^+(2, \mathbb{R})$ are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2, in which case we write $\Gamma_1 \sim \Gamma_2$. Given a subgroup Δ of $GL^+(2, \mathbb{R})$, its commensurator $\widetilde{\Delta} \subset GL^+(2, \mathbb{R})$ is given by

$$\widetilde{\Delta} = \{g \in GL^+(2, \mathbb{R}) \mid g\Delta g^{-1} \sim \Delta\}.$$

If $\Gamma \subset SL(2, \mathbb{R})$ is a discrete subgroup, the double coset $\Gamma\alpha\Gamma$ with $\alpha \in \widetilde{\Gamma}$ has a decomposition of the form

$$(4.2) \qquad \Gamma\alpha\Gamma = \coprod_{i=1}^{s} \Gamma\alpha_i$$

for some $\alpha_i \in GL^+(2, \mathbb{R})$ with $i = 1, \ldots, s$ (see e.g. [19]).

Given an integer λ, we extend the actions of $SL(2, \mathbb{R})$ in (2.4) and (2.7) to those of $GL^+(2, \mathbb{R})$ by setting

$$(f \mid_\lambda \alpha)(z) = \det(\alpha)^{\lambda/2}\mathfrak{J}(\alpha, z)^{-\lambda}f(\alpha z)$$

$$(4.3) \qquad (F \parallel_\lambda \alpha)(z, X) = \det(\alpha)^{\lambda/2}\mathfrak{J}(\alpha, z)^{-\lambda}F(\alpha z, \det(\alpha)^{-1}\mathfrak{J}(\alpha, z)^2(X - \mathfrak{K}(\alpha, z)))$$

for all $z \in \mathcal{H}$, $\alpha \in GL^+(2, \mathbb{R})$, $f \in \mathcal{F}$ and $F(z, X) \in \mathcal{F}_m[X]$. These formulas determine right actions of $GL^+(2, \mathbb{R})$ on $\mathcal{F}$ and $\mathcal{F}_m[X]$, so that we have

$$(f \mid_\lambda \alpha) \mid_\lambda \alpha' = f \mid_\lambda (\alpha\alpha'),$$

$$(4.4) \qquad (F \parallel_\lambda \alpha) \parallel_\lambda \alpha' = F \parallel_\lambda (\alpha\alpha')$$

for all $\alpha, \alpha' \in GL^+(2, \mathbb{R})$.

Let $\alpha \in \widetilde{\Gamma}$ be an element such that the corresponding double coset is as in (4.2). Then the associated Hecke operator

$$(4.5) \qquad T_\lambda(\alpha) : M_\lambda(\Gamma) \to M_\lambda(\Gamma)$$

on modular forms is given as usual by

$$(4.6) \qquad (T_\lambda(\alpha)f)(z) = \sum_{i=1}^{s}(f \mid_\lambda \alpha_i)(z)$$

for all $f \in M_\lambda(\Gamma)$ and $z \in \mathcal{H}$ (cf. [19]). Similarly, given a quasimodular polynomial $F(z, X) \in QP_k^m(\Gamma)$, we set

$$(4.7) \qquad (T_k^P(\alpha)F)(z, X) = \sum_{i=1}^{s}(F \parallel_k \alpha_i)(z, X)$$

for all $z \in \mathcal{H}$.

Proposition 4.1. *For each $\alpha \in \widetilde{\Gamma}$ the polynomial given by (4.7) is independent of the choice of the coset representatives $\alpha_1, \ldots, \alpha_s$, and the map $F \mapsto T_\lambda^P(\alpha)F$ determines the linear endomorphism*

$$T_\lambda^P(\alpha) : QP_\lambda^m(\Gamma) \to QP_\lambda^m(\Gamma).$$

Proof. Suppose that $\{\beta_1, \ldots, \beta_s\}$ is another set of coset representatives with $\beta_i = \gamma_i \alpha_i$ and $\gamma_i \in \Gamma$ for $1 \leq i \leq s$. Given $F(z, X) \in QP_\lambda^m(\Gamma)$, using (4.4), we have

$$(4.8) \qquad \sum_{i=1}^{s}(F \parallel_\lambda \beta_i)(z, X) = \sum_{i=1}^{s}((F \parallel_\lambda \gamma_i) \parallel_\lambda \alpha_i)(z, X)$$
$$= \sum_{i=1}^{s}(F \parallel_\lambda \alpha_i)(z, X),$$

and hence $T_\lambda(\alpha)F$ is independent of the choice of the coset representatives. Since the linearity of the map $F \mapsto T_\lambda(\alpha)F$ is clear, it suffices to show that $T_\lambda(\alpha)(QP_\lambda^m(\Gamma)) \subset QP_\lambda^m(\Gamma)$. From (4.4), (4.7) and (4.8) we see that

$$(T_\lambda(\alpha)F) \parallel_\lambda \gamma = \sum_{i=1}^{s}(F \parallel_\lambda \alpha_i) \parallel_\lambda \gamma = \sum_{i=1}^{s} F \parallel_\lambda (\alpha_i \gamma).$$

However, the set $\{\alpha_1 \gamma, \ldots, \alpha_s \gamma\}$ is another complete set of coset representatives, and therefore we have

$$\sum_{i=1}^{s} F \parallel_\lambda (\alpha_i \gamma) = \sum_{i=1}^{s} F \parallel_\lambda \alpha_i.$$

Thus we see that $(T_\lambda(\alpha)F) \parallel_\lambda \gamma = (T_\lambda(\alpha)F)$; the proposition follows. $\qquad \square$

We now extend the map

$$F(z, X) \mapsto \mathfrak{S}_0(F \|_\xi \gamma)(z) : \mathcal{F}_m[X] \to \mathcal{F}$$

for $\gamma \in SL(2, \mathbb{R})$ to $GL^+(2, \mathbb{R})$ by changing (3.6) to

$$\mathfrak{S}_0(F \|_\xi \alpha)(z) = \sum_{j=0}^{m}(-1)^j(\det \alpha)^{\xi/2 - j} \mathfrak{J}(\alpha, z)^{2j - \xi} \mathfrak{K}(\alpha, z)^j f_j(\alpha z)$$

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for $\alpha \in GL^+(2, \mathbb{R})$. If f is a quasimodular form belonging to $QM_\xi^m(\Gamma)$ and if $\alpha \in \widetilde{\Gamma}$ is an element whose double coset has a decomposition as in (4.2), we set

$$(4.9) \qquad T_\xi^Q(\alpha)f = \sum_{i=1}^s \mathfrak{S}_0((\mathcal{Q}_\xi^m f)\|_\xi \alpha).$$

In other words, $T_\xi^Q(\alpha)$ is deduced from $T_\xi^P(\alpha)$ by conjugation, that is,

$$(4.10) \qquad T_\xi^Q(\alpha) = \mathfrak{S}_0 T_\xi^P(\alpha) Q_\xi^m.$$

So this justifies that $T_\xi^Q(\alpha)f \in QM_\xi^m(\Gamma)$. Thus we obtain the linear endomorphism

$$(4.11) \qquad T_\xi^Q(\alpha) : QM_\xi^m(\Gamma) \to QM_\xi^m(\Gamma)$$

for each $\alpha \in \widetilde{\Gamma}$.

Definition 4.2. The linear endomorphism (4.11) given by (4.9) is the *Hecke operator* on $QM_\xi^m(\Gamma)$ associated to $\alpha \in \widetilde{\Gamma}$.

Remark 4.3. Special types of Hecke operators on quasimodular forms as in Definition 4.2 were considered by Movasati in [20] for $\Gamma = SL(2, \mathbb{Z})$.

Theorem 4.4. *For each $\alpha \in \widetilde{\Gamma}$ the diagram*

$$(4.12) \qquad \begin{array}{ccccc} QM_\xi^m(\Gamma) & \xrightarrow{\mathcal{Q}_\xi^m} & QP_\xi^m(\Gamma) & \xrightarrow{\mathfrak{S}_m} & M_{\xi-2m}(\Gamma) \\ {\scriptstyle T_\xi^Q(\alpha)} \downarrow & & {\scriptstyle T_\xi^P(\alpha)} \downarrow & & \downarrow {\scriptstyle T_{\xi-2m}(\alpha)} \\ QM_\xi^m(\Gamma) & \xrightarrow{\mathcal{Q}_\xi^m} & QP_\xi^m(\Gamma) & \xrightarrow{\mathfrak{S}_m} & M_{\xi-2m}(\Gamma) \end{array}$$

is commutative, where the maps $T_\xi^P(\alpha)$, $T_\xi^Q(\alpha)$ and $T_{\xi-2m}(\alpha)$ are as in (4.7), (4.9) and (4.6), respectively.

Proof. Let α be an element of $\widetilde{\Gamma}$ such that the corresponding double coset has a decomposition as in (4.2). Given $f \in QM_\xi^m(\Gamma)$, using (4.9), we have

$$(\mathcal{Q}_\xi^m \circ T_\xi^Q(\alpha))f = \mathcal{Q}_\xi^m \left(\mathfrak{S}_0 \left(\sum_{i=1}^s (\mathcal{Q}_\xi^m f) \|_\xi \alpha_i \right) \right) = \sum_{i=1}^s (\mathcal{Q}_\xi^m f) \|_\xi \alpha_i,$$

since $\mathcal{Q}_\xi^m = \mathfrak{S}_0^{-1}$ by Proposition 3.6. On the other hand, from (4.7) we obtain

$$(T_\xi^P(\alpha) \circ Q_\xi^m)f = \sum_{i=1}^s (\mathcal{Q}_\xi^m f) \|_\xi \alpha_i;$$

hence it follows that

$$\mathcal{Q}_\xi^m \circ T_\xi^Q(\alpha) = T_\xi^P(\alpha) \circ Q_\xi^m.$$

We now consider a quasimodular polynomial $F(z, X) = \sum_{r=0}^{m} f_r(z) X^r \in QP_\xi^m(\Gamma)$. Using (4.3) and (4.7), we have

$$(\mathfrak{S}_m \circ T_\xi^P(\alpha))F = \mathfrak{S}_m\left(\sum_{i=1}^{s}(F \|_\xi \alpha_i)\right)$$

$$= \sum_{i=1}^{s}(\det \alpha_i)^{\xi/2}\mathfrak{J}(\alpha_i, z)^{-\xi}$$

$$\times \mathfrak{S}_m\left(\sum_{r=0}^{m} f_r(\alpha_i z)(\det \alpha_i)^{-r}\mathfrak{J}(\alpha_i, z)^{2r}(X - \mathfrak{K}(\alpha_i, z))^r\right)$$

$$= \sum_{i=1}^{s}(\det \alpha_i)^{\xi/2-m}\mathfrak{J}(\alpha_i, z)^{2m-\xi}f_m(\alpha_i z)$$

$$= \sum_{i=1}^{s}(\mathfrak{S}_m F)\mid_{\xi-2m}\alpha_i = (T_{\xi-2m}(\alpha) \circ \mathfrak{S}_m)F.$$

Thus we obtain

$$\mathfrak{S}_m \circ T_\xi^P(\alpha) = T_{\xi-2m}(\alpha) \circ \mathfrak{S}_m,$$

and therefore the proof of the theorem is complete. $\qquad\square$

Given $\lambda \in \mathbb{Z}$, we now extend the action of $SL(2, \mathbb{R})$ in (2.5) to that of $GL^+(2, \mathbb{R})$ by setting

(4.13) $\qquad (\Phi \mid_\lambda^J \alpha)(z, X) = \det(\alpha)^{\lambda/2}\mathfrak{J}(\alpha, z)^{-\lambda}e^{-\mathfrak{K}(\alpha, z)X}\Phi(\alpha z, (\det \alpha)\mathfrak{J}(\alpha, z)^{-2}X)$

for $\Phi(z, X) \in \mathcal{F}[[X]]$ and $\alpha \in GL^+(2, \mathbb{R})$. Then it can be shown that

(4.14) $\qquad\qquad\qquad (\Phi \mid_\lambda^J \alpha) \mid_\lambda^J \alpha' = \Phi \mid_\lambda^J (\alpha\alpha')$

for $\alpha, \alpha' \in GL^+(2, \mathbb{R})$.

If $\Phi(z, X) \in \mathcal{J}_\lambda(\Gamma)$ and if the double coset corresponding to an element $\alpha \in \widetilde{\Gamma}$ has a decomposition as in (4.2), we set

(4.15) $\qquad\qquad (T_\lambda^J(\alpha)\Phi)(z, X) = \sum_{i=1}^{s}(\Phi \mid_\lambda^J \alpha_i)(z, X),$

for all $z \in \mathcal{H}$. Then the power series $(T_\lambda^J(\alpha)\Phi)(z, X)$ is independent of the choice of the coset representatives $\alpha_1, \ldots, \alpha_s$, and the map $\Phi \mapsto T_\lambda^J(\alpha)\Phi$ determines the linear endomorphism

$$T_\lambda^J(\alpha) : \mathcal{J}_\lambda(\Gamma) \to \mathcal{J}_\lambda(\Gamma).$$

Theorem 4.5. *If Π_m^δ is as in (2.9), for each $\alpha \in \widetilde{\Gamma}$ the diagram*

$$\begin{array}{ccc}
\mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\ \Pi_m^\delta\ } & QP_{\lambda+2m+2\delta}^m(\Gamma) \\
{\scriptstyle T_\lambda^J(\alpha)}\downarrow & & \downarrow{\scriptstyle T_{\lambda+2m+2\delta}^P(\alpha)} \\
\mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\ \Pi_m^\delta\ } & QP_{\lambda+2m+2\delta}^m(\Gamma),
\end{array}$$

is commutative, where the Hecke operators $T_{\lambda+2m+2\delta}^P(\alpha)$ and $T_\lambda^J(\alpha)$ are as in (4.7) and (4.15), respectively.

Proof. Let α be an element of $\widetilde{\Gamma}$ such that the corresponding double coset has a decomposition as in (4.2). Using (4.7) and (4.15) as well as the extension of (2.13) to $GL^+(2, \mathbb{R})$, we obtain

$$
\begin{aligned}
(\Pi_m^\delta(T_\lambda^J(\alpha)\Phi))(z, X) &= \sum_{i=1}^s \Pi_m^\delta(\Phi \mid_\lambda^J \alpha_i)(z, X) \\
&= \sum_{i=1}^s (\Pi_m^\delta \Phi) \parallel_{\lambda+2m+2\delta} \alpha_i)(z, X) \\
&= ((T_{\lambda+2m+2\delta}^P(\alpha)(\Pi_m^\delta \Phi))(z, X)
\end{aligned}
$$

for all $\Phi(z, X) \in \mathcal{J}_\lambda(\Gamma)_\delta$, which shows the commutativity of the given diagram. $\qquad\square$

Given a modular form $f \in M_{2w+\lambda}(\Gamma)$, it is known that the formal power series

$$
(4.16) \qquad \Phi_f(z, X) = \sum_{\ell=0}^\infty \frac{f^{(\ell+\delta-w)}(z)}{(\ell - w + \delta)!(\ell + w + \delta + \lambda - 1)!} X^{\ell+\delta}
$$

with $p! = 0$ for $p < 0$ is a Jacobi-like form belonging to $\mathcal{J}_\lambda(\Gamma)_\delta$(cf. [8], [25]). Thus the map $f \mapsto \Phi_f$ determines a lifting

$$
(4.17) \qquad \mathcal{L}_{w,\lambda}^X : M_{2w+\lambda}(\Gamma) \to \mathcal{J}_\lambda(\Gamma)_\delta
$$

of modular forms to Jacobi-like forms known as the Cohen-Kuznetsov lifting (see [7]).

Let $\partial = d/dz$ be the derivative operator on $\mathcal{F}$. If h is a modular form belonging to $M_k(\Gamma)$, then it is known that $\partial^\ell h$ with $\ell \geq 1$ is a quasimodular form belonging to $QM_{k+2\ell}^\ell(\Gamma)$ (see e.g. [18]).

Theorem 4.6. *Let f be a modular form belonging to $M_{2w+\lambda}(\Gamma)$ for some $w \in \mathbb{Z}$.*
(i) The quasimodular form $\partial^{m-w+\delta} f \in QM_{\lambda+2m+2\delta}^{m-w+\delta}(\Gamma)$ satisfies

$$
(4.18) \qquad \frac{\partial^{m-w+\delta} f}{(m - w + \delta)!(m + w + \delta + \lambda - 1)!} = (\mathfrak{S}_0 \circ \Pi_m^\delta \circ \mathcal{L}_{w,\lambda}^X)f.
$$

(ii) If $\alpha \in \widetilde{\Gamma}$, then we have

$$
(4.19) \qquad \partial^{m-w+\delta}(T_{2w+\lambda}(\alpha)f) = T_{\lambda+2m+2\delta}^Q(\alpha)\partial^{m-w+\delta} f,
$$

where $T_{2w+\lambda}(\alpha)$ and $T_{\lambda+2m+2\delta}^Q(\alpha)$ are the Hecke operators in (4.5) and (4.9), respectively.

Proof. If $\mathcal{L}_{w,\lambda}^X$ is as in (4.17), we obtain the map

$$
\Pi_m^\delta \circ \mathcal{L}_{w,\lambda}^X : M_{2w+\lambda}(\Gamma) \to QP_{\lambda+2m+2\delta}^m(\Gamma)
$$

given by

$$
((\Pi_m^\delta \circ \mathcal{L}_{w,\lambda}^X)f)(z, X) = \sum_{r=0}^m \frac{f^{(m-r+\delta-w)}(z)}{(m - r - w + \delta)!(m - r + w + \delta + \lambda - 1)!} \frac{X^r}{r!}
$$

for all $f \in M_{2w+\lambda}(\Gamma)$. Thus we obtain (4.18) by considering the coefficient of X^0 in this relation. On the other hand, given $\alpha \in \widetilde{\Gamma}$ and $f \in M_{2w+\lambda}(\Gamma)$, it is known (cf. [16]) that

$$
\mathcal{L}_{w,\lambda}^X(T_\lambda(\alpha)f) = T_\lambda^J(\alpha)(\mathcal{L}_{w,\lambda}^X f).
$$

Hence (ii) follows from this and Theorem 4.5. $\qquad\square$

Theorem 4.6 shows that the diagram

$$\begin{array}{ccc}
M_{2w+\lambda}(\Gamma) & \xrightarrow{\partial^{m-w+\delta}} & QM^m_{\lambda+2m+2\delta}(\Gamma) \\
\scriptstyle{T_{2w+\lambda}(\alpha)}\downarrow & & \downarrow\scriptstyle{T^Q_{\lambda+2m+2\delta}(\alpha)} \\
M_{2w+\lambda}(\Gamma) & \xrightarrow{\partial^{m-w+\delta}} & QM^m_{\lambda+2m+2\delta}(\Gamma)
\end{array}$$

is commutative for each $\alpha \in \widetilde{\Gamma}$.

5. Liftings of modular forms to quasimodular forms

There is a natural surjective map from quasimodular polynomials to modular forms sending a quasimodular polynomial to its leading coefficient. In this section we construct a lifting of a modular form to a quasimodular form whose leading coefficient coincides with the given modular form.

If $\mathfrak{S}_m : \mathcal{F}_m[X] \to \mathcal{F}$ and $\widehat{\mathfrak{S}}^\delta : \mathcal{F}[[X]] \to \mathcal{F}$ are as in (2.10), from (3.8) and Definition 3.9 we see that

$$\widehat{\mathfrak{S}}^\delta(\mathcal{J}_\lambda(\Gamma)_\delta) \subset M_{\lambda+2\delta}(\Gamma), \quad \mathfrak{S}_m(QP^m_\xi(\Gamma)) \subset M_{\xi-2m}(\Gamma).$$

Hence the rows in (2.12) induce the short exact sequences of the form

$$0 \to \mathcal{J}_\lambda(\Gamma)_{\delta+1} \xhookrightarrow{\widehat{\iota}} \mathcal{J}_\lambda(\Gamma)_\delta \xrightarrow{\widehat{\mathfrak{S}}^\delta} M_{\lambda+2\delta}(\Gamma) \to 0,$$

$$0 \to QP^{m-1}_\xi(\Gamma) \xhookrightarrow{\iota} QP^m_\xi(\Gamma) \xrightarrow{\mathfrak{S}_m} M_{\xi-2m}(\Gamma) \to 0.$$

for integers ξ, λ and $\delta > 0$. Furthermore, combining these results with Proposition 3.10 and the isomorphism (3.10), we see that the diagram (2.12) induces the commutative diagram

$$\text{(5.1)}\quad\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J}_\lambda(\Gamma)_{\delta+1} & \xrightarrow{\widehat{\iota}} & \mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\widehat{\mathfrak{S}}^\delta} & M_{\lambda+2\delta}(\Gamma) & \longrightarrow & 0 \\
& & \scriptstyle{\Pi^{\delta+1}_{m-1}}\downarrow & & \downarrow\scriptstyle{\Pi^\delta_m} & & \downarrow\scriptstyle{\mu_m} & & \\
0 & \longrightarrow & QP^{m-1}_{\lambda+2m+2\delta}(\Gamma) & \xrightarrow{\iota} & QP^m_{\lambda+2m+2\delta}(\Gamma) & \xrightarrow{\mathfrak{S}_m} & M_{\lambda+2\delta}(\Gamma) & \longrightarrow & 0 \\
& & \scriptstyle{\mathfrak{S}_0}\downarrow & & \downarrow\scriptstyle{\mathfrak{S}_0} & & \| & & \\
0 & \longrightarrow & QM^{m-1}_{\lambda+2m+2\delta}(\Gamma) & \longrightarrow & QM^m_{\lambda+2m+2\delta}(\Gamma) & \longrightarrow & M_{\lambda+2\delta}(\Gamma) & \longrightarrow & 0
\end{array}$$

in which the three rows are short exact sequences.

Proposition 5.1. *Given integers δ and λ with $\delta > 0$ and a modular form $h \in M_{\lambda+2\delta}(\Gamma)$, the polynomial $(\Xi^{\lambda+2\delta}_m h)(z, X) \in \mathcal{F}_m[X]$ given by*

$$\text{(5.2)}\qquad (\Xi^{\lambda+2\delta}_m h)(z, X) = \sum_{r=0}^m \frac{m!(\lambda + 2\delta - 1)! h^{(m-r)}(z)}{r!(m-r)!(m-r+2\delta+\lambda-1)!} X^r$$

is a quasimodular polynomial belonging to $QP^m_{\lambda+2m+2\delta}(\Gamma)$. Furthermore, we have

$$\text{(5.3)}\qquad \mathfrak{S}_m(\Xi^{\lambda+2\delta}_m h) = h.$$

Proof. Since $h \in M_{\lambda+2\delta}(\Gamma)$, by (4.16) the formal power series $\Phi_h(z, X)$ given by

$$\Phi_h(z, X) = m!(\lambda + 2\delta - 1)! \sum_{\ell=0}^\infty \frac{h^{(\ell)}(z)}{\ell!(\ell+2\delta+\lambda-1)!} X^{\ell+\delta}$$

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is a Jacobi-like form belonging to $\mathcal{J}_\lambda(\Gamma)_\delta$. From this and (2.9) we obtain

$$(\Pi_m^\delta \Phi_h)(z, X) = m!(\lambda + 2\delta - 1)! \sum_{r=0}^{m} \frac{h^{(m-r)}(z)}{r!(m-r)!(m-r+2\delta+\lambda-1)!} X^r$$

$$= (\Xi_m^{\lambda+2\delta} h)(z, X),$$

and it is a quasimodular polynomial belonging to $QP_{\lambda+2m+2\delta}^m(\Gamma)$. On the other hand, we see easily that the coefficient of X^m in this polynomial is equal to h, which shows that $\mathfrak{S}_m(\Xi_m^{\lambda+2\delta} h) = h$. $\qquad\square$

The formula (5.2) determines the linear map

$$\Xi_m^{\lambda+2\delta} : M_{\lambda+2\delta}(\Gamma) \to QP_{\lambda+2m+2\delta}^m(\Gamma),$$

which may be regarded as a lifting from modular forms to quasimodular forms. From (5.3) we see that the short exact sequence in the second row of (5.1) splits. Furthermore, from (4.19) and (5.2) we see that $\Xi_m^{\lambda+2\delta}$ is Hecke equivariant, meaning that the diagram

(5.4)
$$\begin{array}{ccc}
M_{2\delta+\lambda}(\Gamma) & \xrightarrow{\Xi_m^{\lambda+2\delta}} & QP_{\lambda+2m+2\delta}^m(\Gamma) \\
\downarrow{\scriptstyle T_{2\delta+\lambda}(\alpha)} & & \downarrow{\scriptstyle T_{\lambda+2m+2\delta}^P(\alpha)} \\
M_{2\delta+\lambda}(\Gamma) & \xrightarrow{\Xi_m^{\lambda+2\delta}} & QP_{\lambda+2m+2\delta}^m(\Gamma)
\end{array}$$

is commutative for each $\alpha \in \widetilde{\Gamma}$.

We also consider the lifting

$$\widehat{\Xi}_\delta : M_{\lambda+2\delta}(\Gamma) \to \mathcal{J}_\lambda(\Gamma)_\delta$$

from modular forms to Jacobi-like forms defined by

(5.5)
$$(\widehat{\Xi}_\delta f)(z, X) = (\lambda + 2\delta - 1)!(\mathcal{L}_{\delta,\lambda}^X f)(z, X)$$

$$= (\lambda + 2\delta - 1)! \sum_{\ell=0}^{\infty} \frac{f^{(\ell)}(z)}{\ell!(\ell + 2\delta + \lambda - 1)!} X^{\ell+\delta}$$

for $f \in M_{\lambda+2\delta}(\Gamma)$, where $\mathcal{L}_{\delta,\lambda}^X$ is the Cohen-Kuznetsov lifting in (4.17). Then we have

(5.6)
$$\widehat{\mathfrak{S}}^\delta \circ \widehat{\Xi}_\delta(f) = f.$$

It can be shown that the map $\widehat{\Xi}_\delta$ is Hecke equivariant (cf. [16]), so that the diagram

$$\begin{array}{ccc}
M_{2\delta+\lambda}(\Gamma) & \xrightarrow{\widehat{\Xi}_\delta} & \mathcal{J}_\lambda(\Gamma)_\delta \\
\downarrow{\scriptstyle T_{2w+\lambda}(\alpha)} & & \downarrow{\scriptstyle T_\lambda^J(\alpha)} \\
M_{2\delta+\lambda}(\Gamma) & \xrightarrow{\widehat{\Xi}_\delta} & \mathcal{J}_\lambda(\Gamma)_\delta
\end{array}$$

is commutative for each $\alpha \in \widetilde{\Gamma}$. Furthermore, we also obtain the diagram

(5.7)
$$\begin{array}{ccc}
M_{\lambda+2\delta}(\Gamma) & \xrightarrow{\widehat{\Xi}_\delta} & \mathcal{J}_\lambda(\Gamma)_\delta \\
\downarrow{\scriptstyle \mu_m} & & \downarrow{\scriptstyle \Pi_m^\delta} \\
M_{\lambda+2\delta}(\Gamma) & \xrightarrow{\Xi_m^{\lambda+2\delta}} & QP_{\lambda+2m+2\delta}^m(\Gamma)
\end{array}$$

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that is commutative. We note that the linear map

$$\widehat{\Xi}_\delta \circ \mathfrak{S}_m : QP^m_{\lambda+2m+2\delta}(\Gamma) \to \mathcal{J}_\lambda(\Gamma)_\delta$$

satisfies

$$\widehat{\mathfrak{S}}^\delta \circ \widehat{\Xi}_\delta \circ \mathfrak{S}_m = \mathfrak{S}_m,$$

which follows from (5.6) and the relation $\mathfrak{S}_m(QP^m_{\lambda+2m+2\delta}(\Gamma)) \subset M_{\lambda+2\delta}(\Gamma)$.

6. **Liftings of quasimodular forms to Jacobi-like forms**

In Section 5 we considered liftings of modular forms to quasimodular forms as well as Cohen-Kuznetsov liftings of modular forms to Jacobi-like forms that are Hecke equivariant. On the other hand, in Section 3 we studied surjective maps from Jacobi-like forms to quasi-modular polynomials. In this section we construct liftings of quasimodular to Jacobi-like forms corresponding to this surjection that are Hecke equivariant.

It is important to note that the map $\Pi^\delta_m : \mathcal{J}_\lambda(\Gamma)_\delta \to QP^m_{\lambda+2m+2\delta}(\Gamma)$ in (3.15) depends on δ. For example, if ε is an integer with $0 \leq \varepsilon \leq \delta$ a Jacobi like form

$$\Phi(z, X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta$$

can be regarded as an element of $\mathcal{J}_\lambda(\Gamma)_\varepsilon$ by writing it as

$$\Phi(z, X) = \sum_{k=0}^\infty \phi_{k-\delta+\varepsilon}(z) X^{k+\varepsilon}$$

with $\phi_j = 0$ for $j < 0$. Thus, from (2.9) we see that

$$(\Pi^\delta_m \Phi)(z, X) = \sum_{r=0}^m \frac{1}{r!} \phi_{m-r}(z) X^r,$$

$$(\Pi^\varepsilon_m \Phi)(z, X) = \sum_{r=0}^m \frac{1}{r!} \phi_{m-\delta+\varepsilon-r}(z) X^r$$

$$= \sum_{r=0}^{m-\delta+\varepsilon} \frac{1}{r!} \phi_{m-\delta+\varepsilon-r}(z) X^r$$

$$= (\Pi^\varepsilon_{m-\delta+\varepsilon} \Phi)(z, X)$$

Although $(\Pi^\varepsilon_m \Phi)(z, X)$ above belongs to $QP^{m-\delta+\varepsilon}_{\lambda+2m+2\delta}(\Gamma) \subset QP^m_{\lambda+2m+2\delta}(\Gamma)$ and involves only the first $m - \delta + \varepsilon + 1$ coefficients of $\Phi(z, X)$, it is different from $(\Pi^\delta_{m-\delta+\varepsilon} \Phi)(z, X)$ which is given by

$$(\Pi^\delta_{m-\delta+\varepsilon} \Phi)(z, X) = \sum_{r=0}^{m-\delta+\varepsilon} \frac{1}{r!} \phi_{m-r}(z) X^r.$$

In order to construct a lifting from quasimodular polynomials to Jacobi-like forms, as in the next lemma, we first determine a quasimodular polynomial of degree $\leq m - 1$ from one of degree $\leq m$ by using the commutative diagram (5.7). Combining this with the fact that quasimodular forms of degree 0 are modular forms, the desired lifting can be obtained by induction and Cohen-Kuznetsov liftings of modular forms (see Theorem 6.2 below).

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Lemma 6.1. *Let $F(z, X)$ be a quasimodular polynomial belonging to $QP_\xi^m(\Gamma)$, and let m and ξ be integers with $m \geq 1$. Then we have*

(6.1) $$F(z, X) - m!((\Pi_m^\delta \circ \widehat{\Xi}_\delta \circ \mathfrak{S}_m)F)(z, X) \in QP_\xi^{m-1}(\Gamma),$$

where $\mathfrak{S}_m$, $\widehat{\Xi}_\delta$ and Π_m^δ are as in (3.8), (3.15) and (5.5), respectively.

Proof. The proof follow from (5.3) and the commutative diagram (5.7). Indeed, given $F(z, X) \in QP_\xi^m(\Gamma)$, we have $\mathfrak{S}_m F \in M_{\xi-2m}(\Gamma)$, hence we see that

$$((\widehat{\Xi}_\delta \circ \mathfrak{S}_m)F)(z, X) \in \mathcal{J}_{\xi-2m-2\delta}(\Gamma)_\delta.$$

Then the initial term of this Jacobi-like form is equal to $(\mathfrak{S}_m F)(z)$, which implies that the coefficient of X^m in the quasimodular polynomial

$$((\Pi_m^\delta \circ \widehat{\Xi}_\delta \circ \mathfrak{S}_m)F)(z, X) \in QP_\xi^m(\Gamma)$$

is $\frac{1}{m!}(\mathfrak{S}_m F)(z)$. On the other hand, the corresponding coefficient in $F(z, X)$ is equal to $(\mathfrak{S}_m F)(z)$; hence the lemma follows. $\square$

Theorem 6.2. *Given a positive integer m and a quasimodular polynomial $F(z, X) \in QP_\xi^m(\Gamma)$ with $\xi > 2m \geq 2$, there exists a Jacobi-like form $\Phi(z, X) \in \mathcal{J}_{\xi-2m-2\delta}(\Gamma)_\delta$ such that*

$$\Pi_m^\delta \Phi = F$$

for each $\delta \geq 0$.

Proof. We first consider the case where $m = 1$. Let

$$F(z, X) = f_0(z) + f_1(z)X \in QP_\xi^1(\Gamma),$$

so that $f_1 = \mathfrak{S}_1 F \in M_{\xi-2}(\Gamma)$. Using (6.1), we see that there is an element $h \in QP_\xi^0(\Gamma) = M_\xi(\Gamma)$ such that

$$F(z, X) - \Pi_1^\delta(\widehat{\Xi}_\delta f_1)(z, X) = h(z)$$

for all $z \in \mathcal{H}$. However, from (5.6) we see that

$$h = \Pi_0^{\delta+1}(\widehat{\Xi}_{\delta+1} h)$$

with $\widehat{\Xi}_{\delta+1} h \in \mathcal{J}_{\xi-2-2\delta}(\Gamma)_{\delta+1}$. If we write

$$(\widehat{\Xi}_\delta f_1)(z, X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta}, \quad (\widehat{\Xi}_{\delta+1} h)(z, X) = \sum_{k=0}^\infty \psi_k(z) X^{k+\delta+1}$$

with ϕ_k and ψ_k being holomorphic functions for each $k \geq 0$, then we obtain

$$F(z, X) = (\Pi_1^\delta(\widehat{\Xi}_\delta f_1) + \Pi_0^{\delta+1}(\widehat{\Xi}_{\delta+1} h))(z, X) = (\phi_1 + \psi_0)(z) + (\phi_0 + \psi_{-1})(z)X$$

with $\psi_{-1} = 0$. Thus we see that

$$F = \Pi_1^\delta(\widehat{\Xi}_\delta f_1 + \widehat{\Xi}_{\delta+1} h),$$

where

$$(\widehat{\Xi}_\delta f_1 + \widehat{\Xi}_{\delta+1} h)(z, X) = \sum_{k=0}^\infty (\phi_k + \psi_{k-1})(z) X^{k+\delta} \in \mathcal{J}_{\xi-2-2\delta}(\Gamma)_\delta$$

with $\psi_{-1} = 0$; hence the $m = 1$ case follows. We now consider an integer $m > 1$ and assume that the statement in the theorem holds for all positive integers less than m. If $F(z, X) \in QP_\xi^m(\Gamma)$, then by (6.1) there is an element $G(z, X) \in QP_\xi^{m-1}(\Gamma)$ satisfying

$$F - (m!)\Pi_m^\delta(\widehat{\Xi}_\delta(\mathfrak{S}_m F)) = G.$$

Thus by the induction hypothesis there is an element $\Psi(z, X) \in \mathcal{J}_{\xi-2m-2\delta}(\Gamma)_{\delta+1}$ such that

$$\Pi_{m-1}^{\delta+1}\Psi = G.$$

Hence we obtain

$$F = (m!)\Pi_m^\delta(\widehat{\Xi}_\delta(\mathfrak{S}_m F)) + \Pi_{m-1}^{\delta+1}\Psi.$$

Thus, assuming that

$$(m!)(\widehat{\Xi}_\delta(\mathfrak{S}_m F))(z, X) = \sum_{k=0}^\infty \alpha_k(z)X^{k+\delta}, \quad \Psi(z, X) = \sum_{k=0}^\infty \beta_k(z)X^{k+\delta+1}$$

with α_k and β_k holomorphic for each $k \geq 0$, we have

$$(6.2) \qquad F(z, X) = \sum_{r=0}^m \frac{1}{r!}\alpha_{m-r}(z)X^r + \sum_{r=0}^{m-1} \frac{1}{r!}\beta_{m-1-r}(z)X^r$$

$$= \sum_{r=0}^m \frac{1}{r!}(\alpha_{m-r} + \beta_{m-1-r})(z)X^r$$

with $\beta_{-1} = 0$. If we set

$$(6.3) \qquad \Phi(z, X) = \big((m!)\widehat{\Xi}_\delta(\mathfrak{S}_m F) + \Psi\big)(z, X) = \sum_{k=0}^\infty (\alpha_k + \beta_{k-1})(z)X^{k+\delta}$$

with $\beta_{-1} = 0$, then $\Phi(z, X)$ is a Jacobi-like form belonging to $\mathcal{J}_{\xi-2m-2\delta}(\Gamma)_\delta$. Using this and (6.2), we obtain

$$F = \Pi_m^\delta \Phi.$$

Thus the statement is true for m, and the proof of the theorem is complete. $\qquad \square$

Given a quasimodular polynomial $F_m(z, X) \in QP_\xi^m(\Gamma)$ with $\xi > 2m \geq 0$ and $\delta \geq 0$, the proof of Theorem 6.2 provides us with a canonical way of obtaining a Jacobi-like form

$$\Phi_m(z, X) = \sum_{k=0}^\infty \phi_{m,k}(z)X^{k+\delta} \in \mathcal{J}_{\xi-2m-2\delta}(\Gamma)_\delta$$

satisfying $\Pi_m^\delta \Phi_m = F_m$. Indeed, the coefficients of $\Phi_m(z, X)$ can be determined recursively as follows. Using (5.5), we have

$$(m!)(\widehat{\Xi}_\delta(\mathfrak{S}_m F_m))(z, X) = m!(\xi - 2m - 1)! \sum_{k=0}^\infty \frac{(\mathfrak{S}_m F_m)^{(k)}(z)}{k!(k + \xi - 2m - 1)!}X^{k+\delta}.$$

From this and (6.3) we obtain

$$\Phi_m(z, X) = \sum_{k=0}^\infty \left(\frac{m!(\xi - 2m - 1)!}{k!(k + \xi - 2m - 1)!}(\mathfrak{S}_m F_m)^{(k)} + \phi_{m-1,k-1} \right)(z)X^{k+\delta}.$$

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Thus we see that

$$\phi_{m,k} = \frac{m!(\xi - 2m - 1)!}{k!(k + \xi - 2m - 1)!}(\mathfrak{S}_m F_m)^{(k)} + \phi_{m-1,k-1}$$

for all $k, m \geq 1$ with

$$\phi_{0,k} = \frac{F_0^{(k)}}{k!(k + \xi - 1)!};$$

here we note that $F_0(z, X)$ does not contain X and is a function of z only, so that

$$F_0^{(k)}(z, X) = \frac{\partial^k}{\partial z^k} F_0(z, X)$$

for $k \geq 1$.

From the above construction we see that the map $F_m \mapsto \Phi_m$ determines a linear map

(6.4)
$$\mathcal{L}_{m,\xi}^\delta : QP_\xi^m(\Gamma) \to \mathcal{J}_{\xi - 2m - 2\delta}(\Gamma)_\delta,$$

which may be considered as a canonical lifting from quasimodular forms to Jacobi-like forms such that

(6.5)
$$(\Pi_m^\delta \circ \mathcal{L}_{m,\xi}^\delta)F = F$$

for all $F(z, X) \in QP_\xi^m(\Gamma)$. In the case where $m = 0$, we define $\mathcal{L}_{0,\xi}^\delta$ to be the map

$$\widehat{\Xi}_\delta : M_\xi \to \mathcal{J}_{\xi - 2\delta}(\Gamma)_\delta.$$

From the Hecke equivariance of the maps Π_m^δ, $\widehat{\Xi}_\delta$ and $\mathfrak{S}_m$ we obtain the same property for $\mathcal{L}_{m,\xi}^\delta$, so that the diagram

$$
\begin{array}{ccc}
QP_\xi^m(\Gamma) & \xrightarrow{\mathcal{L}_{m,\xi}^\delta} & \mathcal{J}_{\xi - 2m - 2\delta}(\Gamma)_\delta \\
{\scriptstyle T_\xi^Q(\alpha)} \downarrow & & \downarrow {\scriptstyle T_{\xi - 2m - 2\delta}^J(\alpha)} \\
QP_\xi^m(\Gamma) & \xrightarrow{\mathcal{L}_{m,\xi}^\delta} & \mathcal{J}_{\xi - 2m - 2\delta}(\Gamma)_\delta
\end{array}
$$

is commutative for $\alpha \in \widetilde{\Gamma}$. We can also consider the lifting

$$\widetilde{\mathcal{L}}_{m,\xi}^\delta : QM_\xi^m(\Gamma) \to \mathcal{J}_{\xi - 2m - 2\delta}(\Gamma)_\delta$$

of quasimodular forms given by

$$\widetilde{\mathcal{L}}_{m,\xi}^\delta = \mathcal{L}_{m,\xi}^\delta \circ \mathcal{Q}_\xi^\delta,$$

where $\mathcal{Q}_\xi^\delta$ is as in (3.2).

Given integers m, δ and ξ with $\delta, m \geq 0$, there is a short exact sequence of the form

(6.6)
$$0 \to \mathcal{J}_{\xi - 2m - 2\delta}(\Gamma)_{\delta + m} \xrightarrow{\iota} \mathcal{J}_{\xi - 2m - 2\delta}(\Gamma)_\delta \xrightarrow{\Pi_m^\delta} QP_\xi^m(\Gamma) \to 0,$$

where ι is the inclusion map. From the relation (6.5) it follows that the short exact sequence in (6.6) splits. Similarly, there is also a split short exact sequence of the form

$$0 \to \mathcal{J}_{\xi - 2m - 2\delta}(\Gamma)_{\delta + m} \xrightarrow{\iota} \mathcal{J}_{\xi - 2m - 2\delta}(\Gamma)_\delta \xrightarrow{\widetilde{\Pi}_m^\delta} QM_\xi^m(\Gamma) \to 0,$$

where $\widetilde{\Pi}_m^\delta = \mathfrak{S}_0 \circ \Pi_m^\delta$.

Remark 6.3. Since $\mathfrak{S}_m F \in M_{\xi-2m}(\Gamma)$ for $F(z, X) \in QP_\xi^m(\Gamma)$, we see that $\mathfrak{S}_m F = 0$ if $\xi < 2m$. Thus the only case that was not covered in Theorem 6.2 is when $\xi = 2m$. As an example of a quasimodular form for $\xi = 2m$, we can consider the Eisenstein series E_2 given by

$$(6.7) \qquad E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z},$$

where $\sigma_1(n) = \sum_{d|n} d$. Then it is known that E_2 satisfies the relation

$$(6.8) \qquad (E_2 \mid_2 \gamma)(z) = E_2(z) + \frac{6}{i\pi} \mathfrak{K}(\gamma, z)$$

for all $z \in \mathcal{H}$ and $\gamma \in SL(2, \mathbf{Z})$. Hence E_2 is a quasimodular form belonging to $QM_2^1(\Gamma)$ with $\Gamma = SL(2, \mathbf{Z})$, and the associated quasimodular polynomial is given by

$$(\mathcal{Q}_2^1 E_2)(z, X) = E_2(z) + \frac{6}{i\pi} X \in QP_2^1(\Gamma).$$

Using $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, the relation (6.8) can be written in the form

$$E_2(-1/z) = z^2 E_2(z) + \frac{12z}{2\pi i}.$$

Using this and induction, it can be shown that

$$\frac{E_2^{(n)}(-1/z)}{n!(n+1)!} = \sum_{\ell=0}^{n} \frac{E_2^{(\ell)}(z)}{(2\pi i)^{n-\ell}(n-\ell)!\ell!(\ell+1)!} z^{2+\ell+n} + \frac{12 z^{n+1}}{(2\pi i)^{n+1}(n+1)!}$$

$$= \sum_{\ell=-1}^{n} \frac{E_2^{(\ell)}(z)}{(2\pi i)^{(n-\ell)}(n-\ell)!\ell!(\ell+1)!} z^{2+\ell+n},$$

where $(-1)! = 1/12$ and $E_2^{(-1)}(z) = 1$. We now set

$$\Phi_{E_2}(z, X) = \sum_{\ell=-1}^{\infty} \frac{E_2^{(\ell)}(z)}{\ell!(\ell+1)!} (2\pi i X)^\ell.$$

Then it can be shown that

$$\Phi_{E_2}(-1/z, z^{-2} X) = z^2 e^{X/z} \Phi_{E_2}(z, X), \quad \Phi_{E_2}(z+1, X) = \Phi_{E_2}(z, X);$$

hence we have

$$\Phi_{E_2}(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X) = \mathfrak{J}(\gamma, z)^2 e^{-\mathfrak{K}(\gamma, z) X} \Phi_{E_2}(z, X)$$

for $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Since $SL(2, \mathbb{Z})$ is generated by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we see that $\Phi_{E_2}(z, X)$ is a Jacobi-like form belonging to $\mathcal{J}_2(\Gamma)$. On the other hand, we have

$$\Pi_1^0(\Phi_{E_2}) = \mathcal{Q}_2^1 E_2,$$

and therefore the Jacobi-like form $\Phi_{E_2}(z, X)$ is a lifting of the quasimodular polynomial $\mathcal{Q}_2^1 E_2(z, X)$.

Example 6.4. Let $f \in M_w(\Gamma)$ be a modular form of weight $w > 0$. Then its m-th derivative with $m > 0$ is a quasimodular form belonging to $QM_{w+2m}^m(\Gamma)$, and it can be shown that

$$(f^{(m)} \mid_{w+2m} \gamma)(z) = \sum_{k=0}^{m} k! \binom{m}{k} \binom{w+m-1}{k} f^{(m-k)}(z) \mathfrak{K}(\gamma, z)^k$$

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for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$ (see e.g. [18]). Thus, if we denote the corresponding quasimodular polynomial by $F(z, X) \in QP^m_{w+2m}(\Gamma)$, we have

$$F(z, X) = (\mathcal{Q}^m_{w+2m} f^{(m)})(z, X) = \sum_{k=0}^{m} k! \binom{m}{k} \binom{w+m-1}{k} f^{(m-k)}(z) X^k,$$

so that

$$\mathfrak{S}_m F = m! \binom{w+m-1}{m} f \in M_w(\Gamma),$$

(6.9)
$$\widehat{\Xi}_\delta(\mathfrak{S}_m F)(z, X) = (w+m-1)! \sum_{\ell=0}^{\infty} \frac{f^{(\ell)}(z)}{\ell!(\ell+w-1)!} X^{\ell+\delta}.$$

Hence we see that

$$\Pi^\delta_m(\widehat{\Xi}_\delta(\mathfrak{S}_m F))(z, X) = (w+m-1)! \sum_{r=0}^{m} \frac{f^{(m-r)}(z)}{r!(m-r)!(m-r+w-1)!} X^r$$

$$= \frac{1}{m!} F(z, X),$$

and therefore the Jacobi-like form in (6.9) is a lifting of the quasimodular polynomial $(\mathcal{Q}^m_{w+2m} f^{(m)})(z, X)$.

Example 6.5. Let E_2 be the Eisenstein series in (6.7). We consider a modular form $f \in M_w(\Gamma)$ with $w > 0$, and set

$$F(z, X) = f(z)(\mathcal{Q}^1_2 E_2)(z) = f(z) E_2(z) + \frac{6}{i\pi} f(z) X,$$

which is a quasimodular polynomial belonging to $QP^1_{w+2}(\Gamma)$. Then we have

$$\widehat{\Xi}_\delta(\mathfrak{S}_1 F)(z, X) = \frac{6}{i\pi}(w-1)! \sum_{\ell=0}^{\infty} \frac{f^{(\ell)}(z)}{\ell!(\ell+w-1)!} X^{\ell+\delta},$$

so that

$$\Pi^\delta_1(\widehat{\Xi}_\delta(\mathfrak{S}_1 F))(z, X) = \frac{6}{i\pi}(w-1)! \left(\frac{f'(z)}{w!} + \frac{f(z)}{(w-1)!} X \right) = \frac{6}{i\pi w} f'(z) + \frac{6}{i\pi} f(z) X.$$

Thus we obtain

$$F(z, X) - \Pi^\delta_1(\widehat{\Xi}_\delta(\mathfrak{S}_1 F))(z, X) = f(z) E_2(z) - \frac{6}{i\pi w} f'(z) \in QP^0_{w+2}(\Gamma) = M_{w+2}(\Gamma),$$

$$\widehat{\Xi}_{\delta+1}(F - \Pi^\delta_1(\widehat{\Xi}_\delta(\mathfrak{S}_1 F)))(z, X) = (w+1)! \sum_{\ell=0}^{\infty} \frac{1}{\ell!(\ell+w+1)!} \left(f E_2 - \frac{6}{i\pi w} f' \right)^{(\ell)} X^{\ell+\delta}.$$

Hence we see that $F(z, X) = \Pi^\delta_1 \Phi(z, X)$ with

$$\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta+1},$$

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where $\phi_0 = (6/i\pi)f$ and

$$\phi_k = \frac{6(w-1)!}{i\pi k!(k+w-1)!}f^{(k)} + \frac{(w+1)!}{(k-1)!(k+w)!}\left(fE_2 - \frac{6}{i\pi w}f'\right)^{(k-1)}$$

$$= \frac{(w-1)!}{k!(k+w)!}\left(\frac{6}{i\pi}(k+w)f^{(k)} + k!w(w+1)\left(fE_2 - \frac{6}{i\pi w}f'\right)^{(k-1)}\right)$$

for all $k \geq 1$.

The canonical lifting map $\mathcal{L}_{m,\xi}^{\delta}$ in (6.4) can also be used to determine linear maps of quasimodular forms. Indeed, if n is another nonnegative integer, we set

$$\mathfrak{T}_{\xi}^{\delta}(m,n) = \Pi_n^{\delta} \circ \mathcal{L}_{m,\xi}^{\delta}.$$

Then since

$$\Pi_n^{\delta}(\mathcal{J}_{\xi-2m-2\delta})(\Gamma)_{\delta} \to QP_{\xi+2n-2m}^n(\Gamma),$$

we obtain a linear map

$$\mathfrak{T}_{\xi}^{\delta}(m,n) : QP_{\xi}^m(\Gamma) \to QP_{\xi+2n-2m}^n(\Gamma)$$

of quasimodular forms such that $\mathfrak{T}_{\xi}^{\delta}(m,m)$ is the identity map on $QP_{\xi}^m(\Gamma)$.

7. Pseudodifferential operators

Jacobi-like forms for a discrete subgroup $\Gamma \subset SL(2,\mathbb{R})$ are known to be in one-to-one correspondence with Γ-automorphic pseudodifferential operators (see [7]). The noncommutative multiplication operation on the space of pseudodifferential operators given by the Leibniz rule determines a natural Lie algebra structure on the same space. In this section we use the above correspondence and the liftings discussed in Section 6 to construct Lie brackets on the space of quasimodular polynomials.

A pseudodifferential operator over $\mathcal{F}$ is a formal Laurent series in the formal inverse ∂^{-1} of $\partial = d/dz$ with coefficients in $\mathcal{F}$ of the form

$$\Psi(z) = \sum_{k=-\infty}^{u} h_k(z)\partial^k$$

with $u \in \mathbb{Z}$ and $h_k \in \mathcal{F}$ for each $k \leq u$. We denote by $\Psi\mathcal{D}(\mathcal{F})$ the space of all pseudodifferential operators over $\mathcal{F}$. Then the group $SL(2,\mathbb{R})$ acts on $\Psi\mathcal{D}(\mathcal{F})$ on the right by

$$(\Psi \circ \gamma)(z) = \Psi(\gamma z) = \sum_{k=-\infty}^{u} h_k(\gamma z)(\mathfrak{J}(\gamma,z)^2\partial)^k$$

for all $\gamma \in SL(2,\mathbb{C})$, where $\mathfrak{J}(\gamma,z)$ is as in (2.1).

Definition 7.1. Given a discrete subgroup Γ of $SL(2,\mathbb{R})$, an element $\Psi(z) \in \Psi\mathcal{D}(\mathcal{F})$ is an *automorphic pseudodifferential operator for* Γ if it satisfies

$$\Psi \circ \gamma = \Psi$$

for all $\gamma \in \Gamma$. We denote by $\Psi\mathcal{D}(\mathcal{F})^{\Gamma}$ the space of all automorphic pseudodifferential operators for Γ.

If α is an integer, we denote by $\Psi\mathcal{D}(\mathcal{F})_\alpha$ the subspace of $\Psi\mathcal{D}(\mathcal{F})$ consisting of the pseudodifferential operators of the form

$$\sum_{k=0}^\infty \psi_k(z)\partial^{\alpha-k}$$

with $\psi_k \in \mathcal{F}$ for all $k \geq 0$. We also set

$$\Psi\mathcal{D}(\mathcal{F})_\alpha^\Gamma = \Psi\mathcal{D}(\mathcal{F})_\alpha \cap \Psi\mathcal{D}(\mathcal{F})^\Gamma.$$

We now consider an isomorphism between the space of formal power series and that of pseudodifferential operators following the idea of Cohen, Manin, and Zagier in [7]. We shall see below that this isomorphism is equivariant with respect to certain actions of $SL(2,\mathbb{R})$. Given a formal power series

$$F(z,X) = \sum_{k=0}^\infty f_k(z)X^{k+\delta} \in \mathcal{F}[[X]]_\delta$$

and a pseudodifferential operator

$$(7.1) \qquad \Psi(z) = \sum_{k=0}^\infty \psi_k(z)\partial^{-k-\varepsilon} \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}$$

with $\delta, \varepsilon > 0$, we set

$$(7.2) \qquad (\mathcal{I}_\xi^\partial F)(z) = \sum_{k=0}^\infty C_{k+\delta+\xi}f_k(z)\partial^{-k-\delta-\xi},$$

$$(7.3) \qquad (\mathcal{I}_\xi^X \Psi)(z,X) = \sum_{k=0}^\infty C_{k+\varepsilon}^{-1}\psi_k(z)X^{k+\varepsilon-\xi}$$

for each nonnegative integer ξ, where C_η with $\eta > 0$ denotes the integer

$$(7.4) \qquad C_\eta = (-1)^\eta \eta!(\eta-1)!.$$

Then it can be easily seen that

$$(7.5) \qquad (\mathcal{I}_\xi^X \circ \mathcal{I}_\xi^\partial)F = F, \quad (\mathcal{I}_\xi^\partial \circ \mathcal{I}_\xi^X)\Psi = \Psi,$$

and therefore we obtain the maps

$$\mathcal{I}_\xi^\partial : \mathcal{F}[[X]]_\delta \to \Psi\mathcal{D}(\mathcal{F})_{-\delta-\xi}, \quad \mathcal{I}_\xi^X : \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon} \to \mathcal{F}[[X]]_{\varepsilon-\xi}$$

that are complex linear isomorphisms. The following proposition shows that these isomorphisms are $SL(2,\mathbb{R})$-equivariant.

Proposition 7.2. *Let $F(z,X) \in \mathcal{F}[[X]]_\delta$ and $\Psi(z) \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}$, and let ξ be a nonnegative integer. Then we have*

$$(\mathcal{I}_\xi^\partial F) \circ \gamma = \mathcal{I}_\xi^\partial(F \mid_{2\xi}^J \gamma), \quad (\mathcal{I}_\xi^X \Psi) \mid_{2\xi}^J \gamma = \mathcal{I}_\xi^X(\Psi \circ \gamma)$$

for all $\gamma \in SL(2,\mathbb{R})$.

Proof. See [14, Proposition 1.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 7.3. *Let $F(z, X)$, $\Psi(z)$ and ξ be as in Proposition 7.2.*

(i) The formal power series $F(z, X)$ is a Jacobi-like form belonging to $\mathcal{J}_{2\xi}(\Gamma)_\delta$ if and only if $\mathcal{I}_\xi^\partial F \in \Psi\mathcal{D}(\mathcal{F})_{-\delta-\xi}^\Gamma$.

(ii) The pseudodifferential operator $\Psi(z)$ belongs to $\Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}^\Gamma$ if and only if $\mathcal{I}_\xi^X \Psi \in \mathcal{J}_{2\xi}(\Gamma)_{\varepsilon-\xi}$.

Proof. This follows immediately from Proposition 7.2. $\qquad\square$

If $\Psi(z) \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}$ is of the form (7.1), we set

$$(7.6) \qquad ({}^\partial\Pi_m^{-\varepsilon}\Psi)(z, X) = \sum_{r=0}^m \frac{\psi_{m-r}(z)}{r! C_{m-r+\varepsilon}} X^r,$$

where $C_{m-r+\varepsilon}$ is as in (7.4).

Lemma 7.4. *If $\Psi(z) \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}^\Gamma$ and $F(z, X) \in \mathcal{J}_{2\xi}(\Gamma)_{\varepsilon-\xi}$, then we have*

$$(7.7) \qquad {}^\partial\Pi_m^{-\varepsilon}\Psi = \Pi_m^{\varepsilon-\xi}(\mathcal{I}_\xi^X\Psi), \quad \Pi_m^{\varepsilon-\xi}F = {}^\partial\Pi_m^{-\varepsilon}(\mathcal{I}_\xi^\partial F)$$

for all $\xi \in \mathbb{Z}$, where $\Pi_m^{\varepsilon-\xi}$ is as in (3.15). In particular, $({}^\partial\Pi_m^{-\varepsilon}\Psi)(z, X)$ is a quasimodular polynomial belonging to $QP_{2m+2\varepsilon}^m$.

Proof. Let $\Psi(z) \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}$ be as in (7.1), so that $(\mathcal{I}_\xi^X\Psi)(z, X)$ is given by (7.3). Using (2.9), we obtain

$$(\Pi_m^{\varepsilon-\xi}(\mathcal{I}_\xi^X\Psi))(z, X) = \sum_{r=0}^m \frac{1}{r!} C_{m-r+\varepsilon}^{-1} \psi_{m-r}(z) X^r = ({}^\partial\Pi_m^{-\varepsilon}\Psi)(z, X).$$

Since $(\mathcal{I}_\xi^X\Psi)(z, X) \in \mathcal{J}_{2\xi}(\Gamma)_{\varepsilon-\xi}$, we see that $(\Pi_m^{\varepsilon-\xi}(\mathcal{I}_\xi^X\Psi))(z, X)$ belongs to $QP_{2m+2\varepsilon}^m$ by Proposition 3.10 and that the first relation in (7.7) holds. The second relation follows from this and Corollary 7.3. $\qquad\square$

By Lemma 7.4 there is a linear map

$${}^\partial\Pi_m^{-\varepsilon} : \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}^\Gamma \to QP_{2m+2\varepsilon}^m$$

such that the diagram

$$\begin{array}{ccc}
\Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}^\Gamma & \xrightarrow{\ \mathcal{I}_\xi^X\ } & \mathcal{J}_{2\xi}(\Gamma)_{\varepsilon-\xi} \\
{}^{\partial\Pi_m^{-\varepsilon}}\downarrow & & \downarrow{}^{\Pi_m^{\varepsilon-\xi}} \\
QP_{2m+2\varepsilon}^m(\Gamma) & = & QP_{2m+2\varepsilon}^m(\Gamma)
\end{array}$$

is commutative. We also see that there is a short exact sequence of the form

$$(7.8) \qquad 0 \to \Psi\mathcal{D}(\mathcal{F})_{m-\varepsilon}^\Gamma \to \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}^\Gamma \xrightarrow{\ {}^\partial\Pi_m^{-\varepsilon}\ } QP_{2m+2\varepsilon}^m(\Gamma) \to 0.$$

Similarly, there is also a split short exact sequence of the form

$$(7.9) \qquad 0 \to \Psi\mathcal{D}(\mathcal{F})_{m-\varepsilon}^\Gamma \to \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon}^\Gamma \xrightarrow{\ {}^\partial\widetilde{\Pi}_m^{-\varepsilon}\ } QM_{2m+2\varepsilon}^m(\Gamma) \to 0,$$

where ${}^\partial\widetilde{\Pi}_m^\delta = \mathfrak{S}_0 \circ {}^\partial\Pi_m^\delta$. If

$$\mathcal{L}_{m,2m+2\varepsilon}^\delta : QP_{2m+2\varepsilon}^m(\Gamma) \to \mathcal{J}_{2\varepsilon-2\delta}(\Gamma)_\delta$$

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is as in (6.4), we set

$$(7.10) \qquad {}^{\partial}\mathcal{L}^{\delta}_{m,2m+2\varepsilon} = \mathcal{I}^{\partial}_{\varepsilon-\delta} \circ \mathcal{L}^{\delta}_{m,2m+2\varepsilon} : QP^{m}_{2m+2\varepsilon}(\Gamma) \to \Psi\mathcal{D}(\mathcal{F})^{\Gamma}_{-\varepsilon}$$

Using (6.5), (7.7) and (7.10), we have

$$({}^{\partial}\Pi^{-\varepsilon}_{m} \circ {}^{\partial}\mathcal{L}^{\delta}_{m,2m+2\varepsilon})F = (\Pi^{\delta}_{m} \circ \mathcal{I}^{X}_{\varepsilon-\delta}) \circ (\mathcal{I}^{\partial}_{ve-\delta} \circ \mathcal{L}^{\delta}_{m,2m+2\varepsilon})F$$
$$= (\Pi^{\delta}_{m} \circ \mathcal{L}^{\delta}_{m,2m+2\varepsilon})F = F$$

for all $F(z,X) \in QP^{m}_{2m+2\varepsilon}(\Gamma)$. Thus the map ${}^{\partial}\mathcal{L}^{\delta}_{m,2m+2\varepsilon}$ is a lifting of quasimodular forms to automorphic pseudodifferential operators, and both of the short exact sequences (7.8) and (7.9) split.

The noncommutative multiplication operation in $\Psi\mathcal{D}(\mathcal{F})$ defined by the Leibniz rule determines the Lie bracket

$$[\ ,\ ]^{\partial} : \Psi\mathcal{D}(\mathcal{F}) \times \Psi\mathcal{D}(\mathcal{F}) \to \Psi\mathcal{D}(\mathcal{F}),$$

on the same space given by

$$(7.11) \qquad [\Psi(z), \Phi(z)]^{\partial} = \Psi(z)\Phi(z) - \Phi(z)\Psi(z)$$

for all $\Psi(z), \Phi(z) \in \Psi\mathcal{D}(\mathcal{F})$, which provides $\Psi\mathcal{D}(\mathcal{F})$ with a structure of a complex Lie algebra.

Given $\delta_1, \delta_2 > 0$ and pseudodifferential operators $\Psi(z) \in \Psi\mathcal{D}(\mathcal{F})_{-\delta_1}$ and $\Phi(z) \in \Psi\mathcal{D}(\mathcal{F})_{-\delta_2}$ of the form

$$\Psi(z) = \sum_{k=0}^{\infty} \psi_k(z)\partial^{-k-\delta_1}, \quad \Phi(z) = \sum_{k=0}^{\infty} \phi_k(z)\partial^{-k-\delta_2},$$

we have

$$\Psi(z)\Phi(z) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{q=0}^{\infty} \binom{-k-\delta_1}{q} \psi_k(z)\phi^{(q)}_{\ell}(z)\partial^{-k-\ell-q-\delta_1-\delta_2}.$$

Changing the indices from k, ℓ, q to r, p, q with $r = k + \ell + q$ and $p = \ell + q$, we obtain

$$\Psi(z)\Phi(z) = \sum_{r=0}^{\infty}\sum_{p=0}^{r}\sum_{q=0}^{p} \binom{-r-\delta_1+p}{q} \psi_{r-p}(z)\phi^{(q)}_{p-q}(z)\partial^{-r-\delta_1-\delta_2}.$$

Hence we see that

$$(7.12) \qquad [\Psi(z), \Phi(z)]^{\partial} = \Psi(z)\Phi(z) - \Phi(z)\Psi(z) = \sum_{r=0}^{\infty} \eta(\Psi, \Phi)_r(z)\partial^{-r-\delta_1-\delta_2},$$

where

$$\eta(\Psi, \Phi)_r(z) = \sum_{p=0}^{r}\sum_{q=0}^{p}\left(\binom{-r-\delta_1+p}{q}\psi_{r-p}(z)\phi^{(q)}_{p-q}(z)\right.$$
$$\left. - \binom{-r-\delta_2+p}{q}\phi_{r-p}(z)\psi^{(q)}_{p-q}(z)\right)$$

for all $r \geq 0$.

If r, p, q, α and β are integers with $r \geq p \geq q$ and $\alpha, \beta > 0$, then we set

$$(7.13) \qquad \Xi^{r,p,q}_{\alpha,\beta} = \frac{(r-p+\alpha)!(r-p+q+\alpha-1)!(p-q+\beta)!(p-q+\beta-1)!}{q!(r+\alpha+\beta)!(r+\alpha+\beta-1)!}.$$

We now consider formal power series $F(z, X) \in \mathcal{F}[[X]]_{\delta_1}$ and $G(z, X) \in \mathcal{F}[[X]]_{\delta_2}$ given by

$$(7.14) \qquad F(z, X) = \sum_{k=0}^{\infty} f_k(z) X^{k+\delta_1}, \quad G(z, X) = \sum_{k=0}^{\infty} g_k(z) X^{k+\delta_2},$$

and define the associated formal power series $[F(z, X), G(z, X)]^X$ by

$$(7.15) \quad [F(z, X), G(z, X)]^X = \sum_{r=0}^{\infty} \sum_{p=0}^{r} \sum_{q=0}^{p} \left(\Xi_{\delta_1+\xi_1, \delta_2+\xi_2}^{r,p,q} f_{r-p}(z) g_{p-q}^{(q)}(z) \right.$$

$$\left. - \Xi_{\delta_2+\xi_2, \delta_1+\xi_1}^{r,p,q} g_{r-p}(z) f_{p-q}^{(q)}(z) \right) X^{r+\delta_1+\delta_2}.$$

Lemma 7.5. *The formula* (7.15) *determines a bilinear map*

$$[\cdot, \cdot]^X : \mathcal{F}[[X]]_{\delta_1} \times \mathcal{F}[[X]]_{\delta_2} \to \mathcal{F}[[X]]_{\delta_1+\delta_2}$$

of formal power series satisfying

$$(7.16) \qquad \mathcal{I}_{\xi_1+\xi_2}^{\partial} \left([F(z, X), G(z, X)]^X \right) = [\mathcal{I}_{\xi_1}^{\partial}(F(z, X)), \mathcal{I}_{\xi_2}^{\partial}(G(z, X))]^{\partial}$$

for $F(z, X) \in \mathcal{F}[[X]]_{\delta_1}$, $G(z, X) \in \mathcal{F}[[X]]_{\delta_2}$ *and* $\xi_1, \xi_2 \geq 0$.

Proof. Let $F(z, X) \in \mathcal{F}[[X]]_{\delta_1}$ and $G(z, X) \in \mathcal{F}[[X]]_{\delta_2}$ be as in (7.14). Then by (7.2) the pseudodifferential operators $\mathcal{I}_{\xi_1}^{\partial}(F(z, X))$ and $\mathcal{I}_{\xi_2}^{\partial}(G(z, X))$ are given by

$$\mathcal{I}_{\xi_1}^{\partial}(F(z, X)) = \sum_{k=0}^{\infty} C_{k+\delta_1+\xi_1} f_k(z) \partial^{-k-\delta_1-\xi_1},$$

$$\mathcal{I}_{\xi_2}^{\partial}(G(z, X)) = \sum_{k=0}^{\infty} C_{k+\delta_2+\xi_2} g_k(z) \partial^{-k-\delta_2-\xi_2}$$

for $\xi_1, \xi_2 \geq 0$. From these relations and (7.12) we obtain

$$[\mathcal{I}_{\xi_1}^{\partial}(F(z, X)), \mathcal{I}_{\xi_2}^{\partial}(G(z, X))]^{\partial}$$

$$= \sum_{r=0}^{\infty} \sum_{p=0}^{r} \sum_{q=0}^{p} \left(\binom{-r - \delta_1 - \xi_1 + p}{q} C_{r-p+\delta_1+\xi_1} C_{p-q+\delta_2+\xi_2} f_{r-p}(z) g_{p-q}^{(q)}(z) \right.$$

$$\left. - \binom{-r - \delta_2 - \xi_2 + p}{q} C_{r-p+\delta_2+\xi_2} C_{p-q+\delta_1+\xi_1} g_{r-p}(z) f_{p-q}^{(q)}(z) \right) \partial^{-r-2\delta-2\xi}.$$

Using this and (7.3), we have

$$\mathcal{I}_{\xi}^{X}[\mathcal{I}_{\xi}^{\partial}(F(z, X)), \mathcal{I}_{\xi}^{\partial}(G(z, X))]^{\partial}$$

$$= \sum_{r=0}^{\infty} \sum_{p=0}^{r} \sum_{q=0}^{p} \left(\binom{-r - \delta_1 - \xi_1 + p}{q} C_{r-p+\delta_1+\xi_1} C_{p-q+\delta_2+\xi_2} f_{r-p}(z) g_{p-q}^{(q)}(z) \right.$$

$$\left. - \binom{-r - \delta_2 - \xi_2 + p}{q} C_{r-p+\delta_2+\xi_2} C_{p-q+\delta_1+\xi_1} g_{r-p}(z) f_{p-q}^{(q)}(z) \right) \frac{X^{r+\delta_1+\delta_2}}{C_{r+\delta_1+\delta_2+\xi_1+\xi_2}},$$

which can be easily seen to coincide with the right hand side of (7.15). $\qquad \square$

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Corollary 7.6. *If* $[\cdot,\cdot]^\partial$ *is as in* (7.11), *then we have*

$$\tag{7.17} \mathcal{I}^X_{\xi_1+\xi_2}\big([\Psi(z),\Phi(z)]^\partial\big) = [\mathcal{I}^X_{\xi_1}(\Psi(z)), \mathcal{I}^X_{\xi_2}(\Phi(z))]^X$$

for $\Psi(z) \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon_1}$, $\Phi \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon_2}$ *and* $\xi_1, \xi_2 \geq 0$.

Proof. Given $\Psi(z) \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon_1}$ and $\Phi \in \Psi\mathcal{D}(\mathcal{F})_{-\varepsilon_2}$, using (7.5) and (7.16), we have

$$[\Psi(z), \Phi(z)]^\partial = [(\mathcal{I}^\partial_{\xi_1} \circ \mathcal{I}^X_{\xi_1})(\Psi(z)), (\mathcal{I}^\partial_{\xi_2} \circ \mathcal{I}^X_{\xi_2})(\Phi(z))]^\partial$$
$$= \mathcal{I}^\partial_{\xi_1+\xi_2}[\mathcal{I}^X_{\xi_1}(\Psi(z)), \mathcal{I}^X_{\xi_2}(\Phi(z))]^X.$$

Thus we obtain (7.17) by applying $\mathcal{I}^X_{\xi_1+\xi_2}$ to this relation and using (7.5) again. $\qquad\square$

For each $i \in \{1,2\}$ we consider a quasimodular polynomial given by

$$\tag{7.18} F_i(z,X) = \sum_{r=0}^{m} f_{i,r}(z)X^r \in QP^m_{2\xi_i}(\Gamma),$$

and set

$$\tag{7.19} [F_1(z,X), F_2(z,X)]^Q = \sum_{r=0}^{m}\sum_{p=0}^{m-r}\sum_{q=0}^{p} \frac{(r+p)!(m-p+q)!}{r!}$$

$$\times \left(\Xi^{m-r,p,q}_{\delta_1+\xi_1,\delta_2+\xi_2} f_{1,r+p}(z) f^{(q)}_{2,m-p+q}(z) \right.$$

$$\left. - \Xi^{m-r,p,q}_{\delta_2+\xi_2,\delta_1+\xi_1} f_{2,r+p}(z) f^{(q)}_{1,m-p+q}(z) \right) X^r,$$

where the coefficients $\Xi^{*,*,*}_{*,*}$ are as in (7.13). Let $\widehat{\mathcal{F}}_m$ be the graded complex vector space

$$\widehat{\mathcal{F}}_m = \bigoplus_{\ell \geq 0} QP^m_\ell(\Gamma)$$

for $m \geq 0$. The next theorem shows that $\widehat{\mathcal{F}}_m$ has the structure of a complex Lie algebra.

Theorem 7.7. *The bilinear map* $[\cdot,\cdot]^Q$ *given by* (7.19) *is a Lie bracket on the space* $\widehat{\mathcal{F}}_m$ *compatible with the Lie bracket* $[\cdot,\cdot]^X$ *on Jacobi-like forms given by* (7.15), *meaning that the diagram*

$$
\begin{array}{ccc}
\mathcal{J}_{2(\xi_1-m-\delta_1)}(\Gamma)_{\delta_1} \times \mathcal{J}_{2(\xi_2-m-\delta_2)}(\Gamma)_{\delta_2} & \xrightarrow{\;[\cdot,\cdot]^X\;} & \mathcal{J}_{2(\xi_1+\xi_2-2m-\delta_1-\delta_2)}(\Gamma)_{\delta_1+\delta_2} \\[4pt]
{\scriptstyle (\Pi^{\delta_1}_m, \Pi^{\delta_1}_m)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Pi^{\delta_1+\delta_2}_m} \\[4pt]
QP^m_{2\xi_1}(\Gamma) \times QP^m_{2\xi_2}(\Gamma) & \xrightarrow{\;[\cdot,\cdot]^Q\;} & QP^m_{2\xi_1+2\xi_2-2m}(\Gamma)
\end{array}
$$

is commutative.

Proof. Let $F_i(z,X)$ with $1 \leq i \leq 2$ be as in (7.18), and assume that

$$\mathcal{L}^{\delta_i}_{m,2\xi_i} F_i(z,X) = \Phi_i(z,X) = \sum_{k=0}^{\infty} \phi_{i,k}(z)X^{k+\delta_i} \in \mathcal{J}_{\lambda_i-2m-2\delta_i}(\Gamma)_{\delta_i}$$

with $\Pi^{\delta_i}_m(\mathcal{L}^{\delta_i}_{m,2\xi_i} F_i) = F_i$, where $\mathcal{L}^{\delta_i}_{m,2\xi_i}$ is as in (6.4). Then we have

$$\tag{7.20} \phi_{i,r} = (m-r)! f_{i,m-r}$$

for each $r \in \{0, 1, \dots m\}$. From (7.15) we see that

$$[\Phi_1(z, X), \Phi_2(z, X)]^X = \sum_{r=0}^{\infty} \sum_{p=0}^{r} \sum_{q=0}^{p} \left(\Xi_{\delta_1+\xi_1, \delta_2+\xi_2}^{r,p,q} \phi_{1,r-p}(z) \phi_{2,p-q}^{(q)}(z) \right.$$

$$\left. - \Xi_{\delta_2+\xi_2, \delta_1+\xi_1}^{r,p,q} \phi_{2,r-p}(z) \phi_{1,p-q}^{(q)}(z) \right) X^{r+\delta_1+\delta_2},$$

which is a Jacobi-like form belonging to $\mathcal{J}_{2\xi_1+2\xi_2}(\Gamma)_{\delta_1+\delta_2}$. Thus, using (7.20), we obtain

$$\Pi_m^{\delta_1+\delta_2}[\Phi_1(z, X), \Phi_2(z, X)]^X = \sum_{r=0}^{m} \sum_{p=0}^{m-r} \sum_{q=0}^{p} \frac{1}{r!} \left(\Xi_{\delta_1+\xi_1, \delta_2+\xi_2}^{m-r,p,q} \phi_{1,m-r-p}(z) \phi_{2,p-q}^{(q)}(z) \right.$$

$$\left. - \Xi_{\delta_2+\xi_2, \delta_1+\xi_1}^{m-r,p,q} \phi_{2,m-r-p}(z) \phi_{1,p-q}^{(q)}(z) \right) X^r$$

$$= [F_1(z, X), F_2(z, X)]^Q,$$

and it is a quasimodular polynomial belonging to $QP_{2\xi_1+2\xi_2-2m}^m(\Gamma)$. In particular, we have

$$[F_1(z, X), F_2(z, X)]^Q = \Pi_m^{\delta_1+\delta_2}[\mathcal{L}_{m,2\xi_1}^{\delta_1} F_1(z, X), \mathcal{L}_{m,2\xi_2}^{\delta_2} F_2(z, X)]^X.$$

We now consider an element

$$\Psi_i(z, X) = \sum_{k=0}^{\infty} \psi_{i,k}(z) X^{k+\delta_i} \in \mathcal{J}_{\lambda_i-2m-2\delta_i}(\Gamma)_{\delta_i}$$

for each $i \in \{1, 2\}$, and let

$$G_i(z, X) = \Pi_m^{\delta_i} \Psi_i(z, X).$$

If we set

$$[\mathcal{L}_{m,2\xi_1}^{\delta_1} G_1(z, X), \mathcal{L}_{m,2\xi_2}^{\delta_2} G_2(z, X)]^X = \sum_{k=0}^{\infty} \alpha_k(z) X^{k+\delta_i+\delta_i},$$

$$[\Psi_1(z, X), \Psi_2(z, X)]^X = \sum_{k=0}^{\infty} \beta_k(z) X^{k+\delta_i+\delta_i},$$

then from (7.15) we see that $\alpha_k = \beta_k$ for each $k \in \{0, 1, \dots, m\}$; hence we obtain

$$\Pi_m^{\delta_1+\delta_2}[\mathcal{L}_{m,2\xi_1}^{\delta_1} G_1(z, X), \mathcal{L}_{m,2\xi_2}^{\delta_2} G_2(z, X)]^X = \Pi_m^{\delta_1+\delta_2}[\Psi_1(z, X), \Psi_2(z, X)]^X.$$

Thus we have

$$[\Pi_m^{\delta_1} \Psi_1(z, X), \Pi_m^{\delta_2} \Psi_2(z, X)]^Q = \Pi_m^{\delta_1+\delta_2}[\Psi_1(z, X), \Psi_2(z, X)]^X,$$

which proves the theorem. $\qquad \square$

Remark 7.8. From (7.19) we see that the coefficient of X^m in $[F_1(z, X), F_2(z, X)]^Q$ is given by

$$\mathfrak{S}_m[F_1(z, X), F_2(z, X)]^Q = \Xi_{\delta_1+\xi_1, \delta_2+\xi_2}^{0,0,0} f_{1,m}(z) f_{2,m}(z) - \Xi_{\delta_2+\xi_2, \delta_1+\xi_1}^{0,0,0} f_{2,m}(z) f_{1,m}(z) = 0;$$

hence it follows that

$$[F_1(z, X), F_2(z, X)]^Q \in QP_{2\xi_1+2\xi_2-2m}^{m-1}(\Gamma).$$

On the other hand, the coefficient of X^{m-1} is equal to

$$\mathfrak{S}_{m-1}[F_1(z,X), F_2(z,X)]^Q$$

$$= m!\Big(\Xi^{1,0,0}_{\delta_1+\xi_1,\delta_2+\xi_2} f_{1,m-1}(z)f_{2,m}(z) - \Xi^{1,0,0}_{\delta_2+\xi_2,\delta_1+\xi_1} f_{2,m-1}(z)f_{1,m}(z)\Big)$$

$$+ m!\Big(\Xi^{1,1,0}_{\delta_1+\xi_1,\delta_2+\xi_2} f_{1,m}(z)f_{2,m-1}(z) - \Xi^{1,1,0}_{\delta_2+\xi_2,\delta_1+\xi_1} f_{2,m}(z)f_{1,m-1}(z)\Big)$$

$$+ m(m!)\Big(\Xi^{1,1,1}_{\delta_1+\xi_1,\delta_2+\xi_2} f_{1,m}(z)f'_{2,m}(z) - \Xi^{1,1,1}_{\delta_2+\xi_2,\delta_1+\xi_1} f_{2,m}(z)f'_{1,m}(z)\Big)$$

$$= m(m!)\Big(\Xi^{1,1,1}_{\delta_1+\xi_1,\delta_2+\xi_2} f_{1,m}(z)f'_{2,m}(z) - \Xi^{1,1,1}_{\delta_2+\xi_2,\delta_1+\xi_1} f_{2,m}(z)f'_{1,m}(z)\Big),$$

and it is a modular form belonging to $M_{2\xi_1+2\xi_2-4m+2}(\Gamma)$ by (3.8).

Corollary 7.9. *The Lie bracket $[\cdot,\cdot]^Q$ on $\widehat{\mathcal{F}}_m$ given by (7.19) is compatible with the Lie bracket $[\cdot,\cdot]^\partial$ on $\Psi\mathcal{D}(\mathcal{F})$ given by (7.11), so that*

$$
\begin{array}{ccc}
\Psi\mathcal{D}(\mathcal{F})^\Gamma_{-\xi_1+m} \times \Psi\mathcal{D}(\mathcal{F})^\Gamma_{-\xi_2+m} & \xrightarrow{[\cdot,\cdot]^\partial} & \Psi\mathcal{D}(\mathcal{F})^\Gamma_{-\xi_1-\xi_2+2m} \\
{\scriptstyle ({}^\partial\Pi_m^{m-\xi_1}, {}^\partial\Pi_m^{m-\xi_2})}\Big\downarrow & & \Big\downarrow{\scriptstyle {}^\partial\Pi_m^{2m-\xi_1-\xi_2}} \\
QP^m_{2\xi_1}(\Gamma) \times QP^m_{2\xi_2}(\Gamma) & \xrightarrow{\quad[\cdot,\cdot]^Q\quad} & QP^m_{2\xi_1+2\xi_2-2m}(\Gamma)
\end{array}
$$

is commutative, where ${}^\partial\Pi_m^*$ is as in (7.6).

Proof. This follows from (7.7), (7.17) and Theorem 7.7, $\qquad\square$

8. Rankin-Cohen brackets on quasimodular forms

Rankin-Cohen brackets for modular forms are well-known (cf. [7]), and similar brackets for Jacobi-like forms were introduced in ([4], [5], [6]) by using the heat operator. Rankin-Cohen brackets on quasimodular forms were also studied by Martin and Royer in [18] for congruence subgroups of $SL(2,\mathbb{Z})$. In this section we construct Rankin-Cohen brackets on quasimodular forms for more general discrete subgroups of $SL(2,\mathbb{R})$ that are compatible with Rankin-Cohen brackets on Jacobi-like forms.

Given $\mu \in \mathbb{R}$, we consider the formal differential operator $\mathfrak{L}_\mu$ on $\mathcal{F}[[X]]$ given by

$$(8.1) \qquad \mathfrak{L}_\mu = \frac{\partial}{\partial z} - \mu\frac{\partial}{\partial X} - X\frac{\partial^2}{\partial X^2},$$

which may be regarded as the radial heat operator in some sense (cf. [15]).

Proposition 8.1. *Given $\mu \in \mathbb{R}$, $\lambda \in \mathbb{Z}$ and a formal power series $\Phi(z,X) \in \mathcal{F}[[X]]$, we have*

$$(8.2) \qquad (\mathfrak{L}_\mu(\Phi)|^J_{\lambda+2}\gamma)(z,X) = \mathfrak{L}_\mu(\Phi|^J_\lambda\gamma)(z,X) + (\lambda-\mu)\mathfrak{K}(\gamma,z)(\Phi|^J_\lambda\gamma)(z,X)$$

for all $\gamma \in SL(2,\mathbb{R})$.

Proof. Let γ be an element of $SL(2,\mathbb{R})$ whose $(2,1)$-entry is c, so that

$$\frac{\partial}{\partial z}\mathfrak{J}(\gamma,z) = c = \mathfrak{J}(\gamma,z)\mathfrak{K}(\gamma,z),$$

where $\mathfrak{J}(\gamma, z)$ and $\mathfrak{K}(\gamma, z)$ are as in (2.1). Given a formal power series $\Phi(z, X) \in \mathcal{F}[[X]]$, using (2.5), we see that

$$
\frac{\partial}{\partial z}(\Phi \mid_\lambda^J \gamma)(z, X) = -\lambda c \mathfrak{J}(\gamma, z)^{-\lambda-1} e^{-\mathfrak{K}(\gamma,z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
+ \mathfrak{J}(\gamma, z)^{-\lambda} \mathfrak{K}(\gamma, z)^2 X e^{-\mathfrak{K}(\gamma,z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
+ \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma,z)X} \mathfrak{J}(\gamma, z)^{-2} \frac{\partial \Phi}{\partial z}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
+ \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma,z)X} (-2c)\mathfrak{J}(\gamma, z)^{-3} X \frac{\partial \Phi}{\partial X}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X),
$$

$$
\frac{\partial}{\partial X}(\Phi \mid_\lambda^J \gamma)(z, X) = -\mathfrak{J}(\gamma, z)^{-\lambda} \mathfrak{K}(\gamma, z) e^{-\mathfrak{K}(\gamma,z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
+ \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma,z)X} \mathfrak{J}(\gamma, z)^{-2} \frac{\partial \Phi}{\partial X}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
= \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma,z)X}
$$
$$
\times \left(-c\mathfrak{J}(\gamma, z)\Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) + \frac{\partial \Phi}{\partial X}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) \right),
$$

$$
\frac{\partial^2}{\partial X^2}(\Phi \mid_\lambda^J \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma,z)X} \left(c^2 \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) \right.
$$
$$
\left. - 2\mathfrak{K}(\gamma, z)\frac{\partial \Phi}{\partial X}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) + \mathfrak{J}(\gamma, z)^{-2}\frac{\partial^2 \Phi}{\partial X^2}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) \right).
$$

From these relations and (8.1), we obtain

$$
\mathfrak{L}_\mu(\Phi \mid_\lambda^J \gamma)(z, X) = \frac{\partial}{\partial z}(\Phi \mid_\lambda^J \gamma)(z, X) - \mu\frac{\partial}{\partial X}(\Phi \mid_\lambda^J \gamma)(z, X) - X\frac{\partial^2}{\partial X^2}(\Phi \mid_\lambda^J \gamma)(z, X)
$$
$$
= \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma,z)X}
$$
$$
\times \left(-\lambda c\mathfrak{J}(\gamma, z)\Phi + \mathfrak{J}(\gamma, z)^2 \mathfrak{K}(\gamma, z)^2 X\Phi + \frac{\partial \Phi}{\partial z} - 2c\mathfrak{J}(\gamma, z)^{-1}X\frac{\partial \Phi}{\partial X} \right.
$$
$$
+ \mu c\mathfrak{J}(\gamma, z)\Phi - \mu\frac{\partial \Phi}{\partial X} - c^2 X\Phi
$$
$$
\left. + 2\mathfrak{K}(\gamma, z)X\frac{\partial \Phi}{\partial X} - \mathfrak{J}(\gamma, z)^{-2}X\frac{\partial^2 \Phi}{\partial X^2} \right)(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
= \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma,z)X} \left(\frac{\partial \Phi}{\partial z} - \mu\frac{\partial \Phi}{\partial X} - \mathfrak{J}(\gamma, z)^{-2}X\frac{\partial^2 \Phi}{\partial X^2} \right)(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
+ (\mu - \lambda)c\mathfrak{J}(\gamma, z)\Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X),
$$

where we used the relation $c = \mathfrak{J}(\gamma, z)\mathfrak{K}(\gamma, z)$. Thus it follows that

$$
(\mathfrak{L}_\mu(\Phi \mid_\lambda^J \gamma))(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma,z)X}(\mathfrak{L}_\mu\Phi)(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
+ (\mu - \lambda)c\mathfrak{J}(\gamma, z)^{-\lambda-1} e^{-\mathfrak{K}(\gamma,z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$
$$
= (\mathfrak{L}_\mu(\Phi) \mid_{\lambda+2}^J \gamma)(z, X) + (\mu - \lambda)\mathfrak{K}(\gamma, z)(\Phi \mid_\lambda^J \gamma)(z, X),
$$

which verifies (8.2). $\qquad\square$

Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$. If $\mu = \lambda$, then (8.2) can be written in the form

$$(\mathfrak{L}_\lambda(\Phi)|_{\lambda+2}^J \gamma)(z, X) = \mathfrak{L}_\lambda(\Phi|_\lambda^J \gamma)(z, X).$$

Thus we see that

$$\mathfrak{L}_\lambda(\mathcal{J}_\lambda(\Gamma)) \subset \mathcal{J}_{\lambda+2}(\Gamma).$$

If $\mu \neq \lambda$, however, the operator $\mathfrak{L}_\mu$ does not carry Jacobi-like forms to Jacobi-like forms.

If $\Phi(z, X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta} \in \mathcal{F}[[X]]_\delta$ with $\delta \geq 0$, then it can be shown that

$$(\mathfrak{L}_\mu^\ell \Phi)(z, X) = \sum_{k=0}^\infty [\Phi]_{\mu,\delta}^{\ell,k}(z) X^{k+\delta},$$

(Note that $\mathfrak{L}_\mu^\ell$ denotes the composition of ℓ-copies of $\mathfrak{L}_\mu$.) where

(8.3)
$$[\Phi]_{\mu,\delta}^{\ell,k}(z) = \sum_{j=1}^\ell (-1)^{\ell-j} \binom{\ell}{j}$$
$$\times \frac{(k+\delta+\ell-j)!(k+\delta+\mu+\ell-j-1)!}{(k+\delta)!(k+\delta+\mu-1)!} \phi_{k+\ell-j}^{(j)}(z)$$

for all $z \in \mathcal{H}$ and $k \geq 0$.

We now introduce bilinear maps on the space $\mathcal{F}[[X]]$ of formal power series by using the operators of the form $\mathfrak{L}_\mu$ with $\mu \in \frac{1}{2}\mathbb{Z}$. Given a nonnegative integer n, elements $\mu_1, \mu_2 \in \frac{1}{2}\mathbb{Z}$ and formal power series

$$\Phi_1(z, X) \in \mathcal{J}_{\lambda_1}(\Gamma)_{\delta_1}, \quad \Phi_2(z, X) \in \mathcal{J}_{\lambda_2}(\Gamma)_{\delta_2},$$

we define the associated formal power series $[\Phi_1, \Phi_2]_{\mu_1,\mu_2,n}^J(z, X) \in \mathcal{F}[[X]]_{\delta_1+\delta_2}$ by

(8.4)
$$[\Phi_1, \Phi_2]_{\mu_1,\mu_2,n}^J(z, X) = \sum_{u=0}^\infty \xi_{\mu_1,\mu_2,n,u}(z) X^{u+\delta_1+\delta_2}$$

with

(8.5)
$$\xi_{\mu_1,\mu_2,n,u}(z) = \sum_{t=0}^u \sum_{\ell=0}^n (-1)^\ell \binom{n+\lambda_1-\mu_1-1}{n-\ell} \binom{n+\lambda_2-\mu_2-1}{\ell}$$
$$\times [\Phi_1]_{\mu_1,\delta_1}^{\ell,t}(z) [\Phi_2]_{\mu_2,\delta_2}^{n-\ell,u-t}(z)$$

for all $z \in \mathcal{H}$ and $u \geq 0$, where $[\Phi_1]_{\mu_1,\delta_1}^{\ell,t}$ and $[\Phi_2]_{\mu_2,\delta_2}^{n-\ell,u-t}$ are as in (8.3). Here we note that the binomial coefficients of the form $\binom{r}{\ell}$ with $r \in \frac{1}{2}\mathbb{Z}$ is given by

$$\binom{r}{\ell} = \frac{r!}{(r-\ell)!\ell!} = \frac{\Gamma(r+1)}{\Gamma(r-\ell+1)\Gamma(\ell+1)},$$

where $\mathbf{\Gamma}$ is the Gamma function.

Proposition 8.2. *If $\Phi_1(z, X)$ and $\Phi_2(z, X)$ are Jacobi-like forms with*

$$\Phi_1(z, X) \in \mathcal{J}_{\lambda_1}(\Gamma)_{\delta_1}, \quad \Phi_2(z, X) \in \mathcal{J}_{\lambda_2}(\Gamma)_{\delta_2},$$

then $[\Phi_1, \Phi_2]_{\mu_1,\mu_2,n}^J(z, X)$ is a Jacobi-like form belonging to $\mathcal{J}_{\lambda_1+\lambda_2+2n}(\Gamma)_{\delta_1+\delta_2}$.

Proof. This was proved in [6] in the case where $\mu_1 = \mu_2 = 1/2$. The general case can be proved in a similar manner. $\qquad\square$

From Proposition 8.2 we obtain the bilinear map

$$(8.6) \qquad [\,,\,]^J_{\mu_1,\mu_2,n} : \mathcal{J}_{\lambda_1}(\Gamma)_{\delta_1} \times \mathcal{J}_{\lambda_2}(\Gamma)_{\delta_2} \to \mathcal{J}_{\lambda_1+\lambda_2+2n}(\Gamma)_{\delta_1+\delta_2},$$

which may be regarded as the n-th Rankin-Cohen bracket for Jacobi-like forms (see [6]).

Given integers λ_1, λ_2, nonnegative integers $\delta_1, \delta_2, m_1, m_2, n$, and quasimodular polynomials

$$(8.7) \quad F_1(z,X) \in QP^{m_1}_{\lambda_1+2(m_1+\delta_1)}(\Gamma) \subset \mathcal{F}_{m_1}[X], \quad F_2(z,X) \in QP^{m_2}_{\lambda_2+2(m_2+\delta_2)}(\Gamma) \subset \mathcal{F}_{m_2}[X],$$

using the lifting map in (6.4), we obtain the Jacobi-like forms

$$(8.8) \qquad \mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)}F_1(z,X) \in \mathcal{J}_{\lambda_1}(\Gamma)_{\delta_1}, \quad \mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)}F_2(z,X) \in \mathcal{J}_{\lambda_2}(\Gamma)_{\delta_2}.$$

If $\mu_1, \mu_2 \in \frac{1}{2}\mathbb{Z}$, we define the bilinear map

$$[\,,\,]^{\lambda_1,\lambda_2,P}_{\delta_1,\delta_2,\mu_1,\mu_2,n} : QP^{m_1}_{\lambda_1+2(m_1+\delta_1)}(\Gamma) \times QP^{m_2}_{\lambda_2+2(m_2+\delta_2)}(\Gamma) \to \mathcal{F}_{m_1+m_2}[X]$$

of polynomials by

$$(8.9) \qquad [F_1, F_2]^{\lambda_1,\lambda_2,P}_{\delta_1,\delta_2,\mu_1,\mu_2,n}(z,X)$$

$$= \sum_{r=0}^{m_1+m_2} \sum_{\ell=0}^{n} \sum_{t=0}^{m_1+m_2-r} \frac{(-1)^\ell}{r!} \binom{n+\lambda_1-\mu_1-1}{n-\ell} \binom{n+\lambda_2-\mu_2-1}{\ell}$$

$$\times [\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)}F_1]^{\ell,t}_{\mu_1,\delta_1}(z)$$

$$\times [\mathcal{L}^{\delta_2}_{m_2,\mu_2+2(m_2+\delta_2)}F_2]^{n-\ell,m_1+m_2-r-t}_{\mu_2,\delta_2}(z)X^r,$$

where the square brackets on the right hand side are as in (8.3).

Proposition 8.3. *The formula* (8.9) *determines a bilinear map*

$$(8.10) \quad [\,,\,]^{\lambda_1,\lambda_2,P}_{\delta_1,\delta_2,\mu_1,\mu_2,n} : QP^{m_1}_{\lambda_1+2(m_1+\delta_1)}(\Gamma) \times QP^{m_2}_{\lambda_2+2(m_2+\delta_2)}(\Gamma) \to QP^{m_1+m_2}_{\lambda_1+\lambda_2+2(m_1+m_2+n+\delta_1+\delta_2)}(\Gamma)$$

of quasimodular polynomials.

Proof. Let $F_1(z,X)$ and $F_2(z,X)$ be quasimodular polynomials as in (8.7). If $[\,,\,]^J_n$ is as in (8.6), then from (8.4) and (8.5) we obtain

$$(8.11) \qquad [\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)}F_1, \mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)}F_2]^J_{\mu_1,\mu_2,n}(z,X)$$

$$= \sum_{u=0}^{\infty} \widetilde{\xi}_u(z)X^{u+\delta_1+\delta_2} \in \mathcal{J}_{\lambda_1+\lambda_2}(\Gamma)_{\delta_1+\delta_2},$$

where

$$(8.12) \qquad \widetilde{\xi}_u(z) = \sum_{t=0}^{u}\sum_{\ell=0}^{n}(-1)^\ell \binom{n+\lambda_1-\mu_1-1}{n-\ell}\binom{n+\lambda_2-\mu_2-1}{\ell}$$

$$\times [\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)}F_1]^{\ell,t}_{\mu_1,\delta_1}(z)[\mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)}F_2]^{n-\ell,u-t}_{\mu_2,\delta_2}(z).$$

Using (6.5), (8.9). (8.11) and (8.12), we see that

$$[F_1, F_2]^{\lambda_1,\lambda_2,P}_{\delta_1,\delta_2,\mu_1,\mu_2,n}(z,X) = \sum_{r=0}^{m_1+m_2} \frac{1}{r!}\widetilde{\xi}_{m_1+m_2-r}(z)X^r$$

$$= \Pi^{\delta_1+\delta_2}_{m_1+m_2}([\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)}F_1, \mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)}F_2]^J_{\mu_1,\mu_2,n}(z,X)),$$

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which is a quasimodular polynomial belonging to

$$QP^{m_1+m_2}_{\lambda_1+\lambda_2+2(m_1+m_2+n+\delta_1+\delta_2)}(\Gamma);$$

hence the theorem follows. $\qquad\square$

The bilinear map (8.10) for quasimodular polynomials determines a corresponding map for quasimodular forms, which may be regarded as a Rankin-Cohen bracket. We now consider a variation of such a bilinear map. Given a nonnegative integer m and quasimodular polynomials $F_1(z,X)$ and $F_2(z,X)$ as in (8.7), by applying the map $\Pi_m^{\delta_1+\delta_2}$ to the left hand side of (8.11) we obtain

$$(8.13) \qquad \Pi_m^{\delta_1+\delta_2}([\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)}F_1, \mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)}F_2]^J_{\mu_1,\mu_2,n}(z,X))$$

$$= \sum_{r=0}^{m} \frac{1}{r!}\widetilde{\xi}_{m-r}(z)X^r$$

$$= \sum_{r=0}^{m}\sum_{\ell=0}^{n}\sum_{t=0}^{m-r+n} \frac{(-1)^\ell}{r!}\binom{n+\lambda_1-\mu_1-1}{n-\ell}\binom{n+\lambda_2-\mu_2-1}{\ell}$$

$$\times [\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)}F_1]^{\ell,t}_{\mu_1,\delta_1}(z)$$

$$\times [\mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)}F_2]^{n-\ell,m-r-t}_{\mu_2,\delta_2}(z)X^r,$$

which is a quasimodular polynomial belonging to

$$QP^{m}_{\lambda_1+\lambda_2+2(m+n+\delta_1+\delta_2)}(\Gamma).$$

Thus, if we set

$$[[F_1,F_2]]^m_n(z,X) = \Pi_m^{\delta_1+\delta_2}([\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)}F_1, \mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)}F_2]^J_{\mu_1,\mu_2,n}(z,X)),$$

we obtain the bilinear map

$$(8.14) \qquad [[\,,\,]]^m_n : QP^{m_1}_{\lambda_1+2(m_1+\delta_1)}(\Gamma) \times QP^{m_2}_{\lambda_2+2(m_2+\delta_2)}(\Gamma) \to QP^{m}_{\lambda_1+\lambda_2+2(m+n+\delta_1+\delta_2)}(\Gamma).$$

If $G(z,X) = \sum_{k=0}^{q} g_k(z)X^k \in \mathcal{F}_q[X]$ with $q \geq 0$, we set

$$(8.15) \qquad [[G]]^{\ell,k}_{\mu,\delta}(z) = \sum_{j=0}^{\ell}(q-k-j)!(-1)^{\ell-j}\binom{\ell}{j}$$

$$\times \frac{(k+\delta+\ell-j)!(k+\delta+\mu-\ell-j-1)!}{(k+\delta)!(k+\delta+\mu-1)!}g^{(j)}_{k+\ell-j}(z)$$

for $z \in \mathcal{H}$, $\delta \geq 0$ and $0 \leq k \leq q$.

Proposition 8.4. *Let $F_1(z,X)$ and $F_2(z,X)$ be as in Proposition 8.3, and let m be a nonnegative integer with $m+2n \leq \min\{m_1,m_2\}$. Then the bilinear map (8.14) is given by*

$$(8.16)$$

$$[[F_1,F_2]]^m_n(z,X) = \sum_{r=0}^{m_1+m_2}\sum_{\ell=0}^{n}\sum_{t=0}^{m_1+m_2-r+n} \frac{(-1)^\ell}{r!}\binom{n+\lambda_1-\mu_1-1}{n-\ell}\binom{n+\lambda_2-\mu_2-1}{\ell}$$

$$\times [[F_1]]^{\ell,t}_{\mu,\delta_1}(z)[[F_2]]^{n-\ell,m-r-t}_{\mu,\delta_2}(z)X^r,$$

and the diagram

$$(8.17) \qquad \begin{array}{ccc} \mathcal{J}_{\lambda_1}(\Gamma)_{\delta_1} \times \mathcal{J}_{\lambda_2}(\Gamma)_{\delta_2} & \xrightarrow{[\,,\,]^J_{\mu_1,\mu_2,n}} & \mathcal{J}_{\lambda_1+\lambda_2+2n}(\Gamma)_{\delta_1+\delta_2} \\ {\scriptstyle (\Pi^{\delta_1}_{m_1},\Pi^{\delta_2}_{m_2})} \downarrow & & \downarrow {\scriptstyle \Pi^{\delta_1+\delta_2}_{m_1+m_2}} \\ QP^{m_1}_{\lambda_1+2(m_1+\delta_1)}(\Gamma) \times QP^{m_2}_{\lambda_2+2(m_2+\delta_2)}(\Gamma) & \xrightarrow{[[\,,\,]]_m} & QP^m_{\lambda_1+\lambda_2+2(m+n+\delta_1+\delta_2)}(\Gamma) \end{array}$$

is commutative.

Proof. If $m + 2n \leq \min\{m_1, m_2\}$, then we see from (8.3) that the functions

$$[\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)} F_1]^{\ell,t}_{\mu_1,\delta_1}(z), \quad [\mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)} F_2]^{n-\ell,m-r-t}_{\mu_2,\delta_2}(z)$$

in (8.13) involve coefficients of X^j only for $j \leq \min\{m_1, m_2\}$ in the Jacobi-like forms

$$\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)} F_1(z, X), \quad \mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)} F_2(z, X).$$

From this and the relations

$$\Pi^{\delta_1}_{m_1} \mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)} F_1(z, X) = F_1(z, X), \quad \Pi^{\delta_2}_{m_2} \mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)} F_2(z, X) = F_2(z, X),$$

it follows that

$$[\mathcal{L}^{\delta_1}_{m_1,\lambda_1+2(m_1+\delta_1)} F_1]^{\ell,t}_{\mu_1,\delta_1} = [[F_1]]^{\ell,t}_{\mu_1,\delta_1},$$
$$[\mathcal{L}^{\delta_2}_{m_2,\lambda_2+2(m_2+\delta_2)} F_2]^{n-\ell,m-r-t}_{\mu_2,\delta_2} = [[F_2]]^{n-\ell,m-r-t}_{\mu_2,\delta_2};$$

hence we obtain (8.16). The commutativity of the diagram (8.17) also follows from the above observations. $\qquad\square$

9. Quasimodular forms of half-integral weight

In this section we modify the definitions of Jacobi-like forms and quasimodular forms to include the half-integral weights. We also describe Hecke operators on spaces of those forms. Throughout this section we assume that $\Gamma = \Gamma_0(4N)$ for some positive integer N.

Let $\theta(z)$ be the theta series given by

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z}$$

for $z \in \mathcal{H}$, and set

$$(9.1) \qquad \widehat{\mathfrak{J}}(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}, \quad \widehat{\mathfrak{K}}(\gamma, z) = 2\widehat{\mathfrak{J}}(\gamma, z)^{-1} \frac{d}{dz} \widehat{\mathfrak{J}}(\gamma, z)$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, it is known that

$$\widehat{\mathfrak{J}}(\gamma, z)^2 = \left(\frac{-1}{d} \right)(cz + d),$$

where $(\frac{\cdot}{\cdot})$ denotes the Legendre symbol. Furthermore, the resulting maps $\widehat{\mathfrak{J}}, \widehat{\mathfrak{K}} : \Gamma \times \mathcal{H} \to \mathbb{C}$ satisfy

$$(9.2) \qquad \widehat{\mathfrak{J}}(\gamma\gamma', z) = \widehat{\mathfrak{J}}(\gamma, \gamma' z)\widehat{\mathfrak{J}}(\gamma', z), \quad \widehat{\mathfrak{K}}(\gamma\gamma', z) = \widehat{\mathfrak{K}}(\gamma', z) + \widehat{\mathfrak{J}}(\gamma', z)^{-2}\widehat{\mathfrak{K}}(\gamma, \gamma' z)$$

for all $\gamma, \gamma' \in \Gamma$ and $z \in \mathcal{H}$.

We recall that $\mathcal{F}$ denotes the ring of holomorphic functions on $\mathcal{H}$ and $\mathcal{F}[[X]]$ is the complex algebra of formal power series in X with coefficients in $\mathcal{F}$. From now on we assume that λ

is a half integer, so that 2λ is an odd integer. Given elements $f \in \mathcal{F}$, $\Phi(z, X) \in \mathcal{F}[[X]]$, and $\gamma \in \Gamma$, we set

$$(9.3) \qquad (f \mid_\lambda \gamma)(z) = \widehat{\mathfrak{J}}(\gamma, z)^{-2\lambda} f(z)$$

$$(9.4) \qquad (\Phi \mid_\lambda^J \gamma)(z, X) = \widehat{\mathfrak{J}}(\gamma, z)^{-2\lambda} e^{-\widehat{\mathfrak{K}}(\gamma, z) X} \Phi(\gamma z, \widehat{\mathfrak{J}}(\gamma, z)^{-2} X)$$

for all $z \in \mathcal{H}$. If γ' is another element of Γ, then by using (9.2) it can be shown that

$$f \mid_\lambda (\gamma \gamma') = (f \mid_\lambda \gamma) \mid_\lambda \gamma', \quad \Phi \mid_\lambda^J (\gamma \gamma') = (\Phi \mid_\lambda^J \gamma) \mid_\lambda^J \gamma',$$

so that the operations $\mid_\lambda$ and $\mid_\lambda^J$ determine right actions of Γ on $\mathcal{F}$ and $\mathcal{F}[[X]]$, respectively.

Given a nonnegative integer m, let $\mathcal{F}_m[X]$ be the complex algebra of polynomials in X over $\mathcal{F}$ of degree at most m as before. If $\gamma \in \Gamma$ and $F(z, X) \in \mathcal{F}_m[X]$, we set

$$(9.5) \qquad (F \parallel_\lambda \gamma)(z, X) = \widehat{\mathfrak{J}}(\gamma, z)^{-2\lambda} F(\gamma z, \widehat{\mathfrak{J}}(\gamma, z)^2 (X - \widehat{\mathfrak{K}}(\gamma, z)))$$

for all $z \in \mathcal{H}$. Then this formula determines a right action $\parallel_\lambda$ of Γ on $\mathcal{F}_m[X]$. Let χ be a character on Γ.

Definition 9.1. Given a half integer λ and a nonnegative integer m, an element $f \in \mathcal{F}$ is a *quasimodular form for Γ of weight λ and depth at most m with character χ* if there are functions $f_0, \ldots, f_m \in \mathcal{F}$ such that

$$(9.6) \qquad (f \mid_\lambda \gamma)(z) = \sum_{r=0}^m \chi(\gamma) f_r(z) \widehat{\mathfrak{K}}(\gamma, z)^r$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $\widehat{\mathfrak{K}}(\gamma, z)$ is as in (9.1) and $\mid_\lambda$ is the operation in (9.3). We denote by $QM_\lambda^m(\Gamma, \chi)$ the space of quasimodular forms for Γ of weight λ and depth m with character χ.

If we denote by $M_\lambda(\Gamma, \chi)$ the space of modular forms for Γ of weight λ and character χ, then we see that

$$QM_\lambda^0(\Gamma, \chi) = M_\lambda(\Gamma, \chi).$$

We also note that $f = f_0$ if f satisfies (9.6).

Let $f \in \mathcal{F}$ be a quasimodular form belonging to $QM_\lambda^m(\Gamma, \chi)$ and satisfying (9.6). Then we define the corresponding polynomial $(\mathcal{Q}_\lambda^m f)(z, X) \in \mathcal{F}_m[X]$ by

$$(9.7) \qquad (\mathcal{Q}_\lambda^m f)(z, X) = \sum_{r=0}^m f_r(z) X^r,$$

so that we obtain the linear map

$$\mathcal{Q}_\lambda^m : QM_\lambda^m(\Gamma, \chi) \to \mathcal{F}_m[X]$$

for each of nonnegative integers m.

Definition 9.2. (i) A *quasimodular polynomial for Γ of weight λ and degree at most m* is an element of $\mathcal{F}_m[X]$ that is Γ-invariant with respect to the right Γ-action in (9.5). We denote by $QP_\lambda^m(\Gamma, \chi)$ the space of all quasimodular polynomials for Γ of weight λ and degree at most m with character χ.

(ii) A formal power series $\Phi(z, X)$ belonging to $\mathcal{F}[[X]]$ is a *Jacobi-like form for Γ of weight λ with character χ* if it satisfies

$$(\Phi \mid_\lambda^J \gamma)(z, X) = \chi(\gamma) \Phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$, where $|_\lambda^J$ is as in (9.4).

We denote by $\mathcal{J}_\lambda(\Gamma, \chi)$ the space of all Jacobi-like forms for Γ of weight λ with character χ, and set

$$\mathcal{J}_\lambda(\Gamma, \chi)_\delta = \mathcal{J}_\lambda(\Gamma, \chi) \cap \mathcal{F}[[X]]_\delta$$

for each nonnegative integer δ.

We now extend the notion of Hecke operators to the half-integral cases. We first recall that $GL^+(2, \mathbb{R})$ acts on $\mathcal{H}$ by linear fractional transformations, and set

$$\mathcal{G} = \{(\alpha, \det(\alpha)^{-1/2}\widehat{\mathfrak{J}}(\alpha, z)) \mid \alpha \in GL^+(2, \mathbb{R})\}.$$

Then $\mathcal{G}$ is a group with respect to the multiplication given by

$$(\alpha, \det(\alpha)^{-1/2}\widehat{\mathfrak{J}}(\alpha, z)) \cdot (\beta, \det(\beta)^{-1/2}\widehat{\mathfrak{J}}(\beta, z)) = (\alpha\beta, \det(\alpha\beta)^{-1/2}\widehat{\mathfrak{J}}(\alpha, \beta(z))\widehat{\mathfrak{J}}(\beta, z)).$$

We shall write $\tilde{\alpha} = (\alpha, \det(\alpha)^{-1/2}\widehat{\mathfrak{J}}(\alpha, z)) \in \mathcal{G}$.

As in the case of $GL^+(2, \mathbb{R})$, two subgroups Γ_1 and Γ_2 of $\mathcal{G}$ are commensurable, or $\Gamma_1 \sim \Gamma_2$, if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2. Given a subgroup Δ of $\mathcal{G}$, the subgroup

$$\widetilde{\Delta} = \{\tilde{g} \in \mathcal{G} \mid g\Delta g^{-1} \sim \Delta\} \subset \mathcal{G}$$

is the commensurator of Δ. If $\mathcal{G}_1 = \{\tilde{\alpha} \in \mathcal{G} \mid \det\alpha = 1\}$ and if Γ is a discrete subgroup of $\mathcal{G}_1$, then the double coset $\Gamma\alpha\Gamma$ with $\tilde{\alpha} \in \widetilde{\Gamma}$ has a decomposition of the form

$$(9.8) \qquad\qquad \Gamma\alpha\Gamma = \coprod_{i=1}^{s} \Gamma\alpha_i$$

for some elements $\alpha_i \in \mathcal{G}$ with $1 \le i \le s$.

Given λ with $2\lambda \in \mathbb{Z}$ odd, we extend the actions of $SL(2, \mathbb{R})$ in (2.4), (2.5) and (2.7) to those of $\mathcal{G}$ by setting

$$(f \mid_\lambda \alpha)(z) = \det(\alpha)^\lambda \widehat{\mathfrak{J}}(\alpha, z)^{-2\lambda} f(\alpha z)$$

$$(9.9) \qquad (\Phi \mid_\lambda^J \alpha)(z, X) = (\det\xi)^\lambda \widehat{\mathfrak{J}}(\alpha, z)^{-2\lambda} e^{-\widehat{\mathfrak{K}}(\alpha,z)} \Phi(\alpha z, (\det\alpha)\widehat{\mathfrak{J}}(\alpha, z)^{-2} X),$$

$$(F \parallel_\lambda \alpha)(z, X) = (\det\alpha)^\lambda \widehat{\mathfrak{J}}(\alpha, z)^{-2\lambda} F(\alpha z, (\det\alpha)^{-1}\widehat{\mathfrak{J}}(\alpha, z)^2(X - \widehat{\mathfrak{K}}(\alpha, z)))$$

for all $z \in \mathcal{H}$, $\tilde{\alpha} \in \mathcal{G}$, $f \in \mathcal{F}$, $\Phi(z, X) \in \mathcal{F}[[X]]$ and $F(z, X) \in \mathcal{F}_m[X]$. Then we have
(9.10)
$$(f \mid_\lambda \alpha) \mid_\lambda \alpha' = f \mid_\lambda (\alpha\alpha'), \quad (\Phi \mid_\lambda^J \alpha) \mid_\lambda^J \alpha' = \Phi \mid_\lambda^J (\alpha\alpha'), \quad (F \parallel_\lambda \alpha) \parallel_\lambda \alpha' = F \parallel_\lambda(\alpha\alpha')$$

for all $\tilde{\alpha}, \tilde{\alpha}'$, and therefore $\mathcal{G}$ acts on $\mathcal{F}$, $\mathcal{F}[[X]]$ and $\mathcal{F}_m[X]$ on the right.

Let $\tilde{\alpha} \in \widetilde{\Gamma} \subset \mathcal{G}$ be an element whose double coset is as in (9.8). Then the associated Hecke operator

$$(9.11) \qquad\qquad T_\lambda : M_\lambda(\Gamma, \chi) \to M_\lambda(\Gamma, \chi)$$

is given as usual by

$$(T_\lambda(\tilde{\alpha})f)(z) = \det(\alpha)^{-\lambda-1} \sum_{i=1}^{s} (\chi(\alpha_i)f \mid_\lambda \alpha_i)(z)$$

for all $f \in M_\lambda(\Gamma, \chi)$ and $z \in \mathcal{H}$. Similarly, given a Jacobi-like form $\Phi(z, X) \in \mathcal{J}_\lambda(\Gamma, \chi)$ and a quasimodular polynomial $F(z, X) \in QP_\lambda^m(\Gamma, \chi)$, we set

$$(9.12) \qquad (T_\lambda^J(\tilde{\alpha})\Phi)(z, X) = \det(\alpha)^{-\lambda-1} \sum_{i=1}^{s} (\Phi \mid_\lambda^J \alpha_i)(z, X),$$

$$(9.13) \qquad (T_\lambda^P(\tilde{\alpha})F)(z, X) = \det(\alpha)^{-\lambda-1} \sum_{i=1}^{s} (F \parallel_\lambda \alpha_i)(z, X)$$

for all $z \in \mathcal{H}$.

Proposition 9.3. *For each $\tilde{\alpha} \in \widetilde{\Gamma}$ the formal power series $(T_\lambda^J(\tilde{\alpha})\Phi)(z, X)$ and the polynomial $(T_\lambda^P(\tilde{\alpha})F)(z, X)$ given by (9.12) and (9.13), respectively, are independent of the choice of the coset representatives $\alpha_1, \ldots, \alpha_s$, and the maps $\Phi \mapsto T_\lambda^J(\tilde{\alpha})\Phi$ and $F \mapsto T_\lambda^P(\tilde{\alpha})F$ determine linear endomorphisms*

$$(9.14) \qquad T_\lambda^J(\tilde{\alpha}) : \mathcal{J}_\lambda(\Gamma, \chi) \to \mathcal{J}_\lambda(\Gamma, \chi), \quad T_\lambda^P(\tilde{\alpha}) : QP_\lambda^m(\Gamma, \chi) \to QP_\lambda^m(\Gamma, \chi).$$

Proof. This can be proved as in the case of the usual Hecke operators for modular forms. $\square$

The endomorphisms $T_\lambda^J(\tilde{\alpha})$ and $T_\lambda^P(\tilde{\alpha})$ are Hecke operators on $\mathcal{J}_\lambda(\Gamma, \chi)$ and $QP_\lambda^m(\Gamma, \chi)$, respectively, for half-integral Jacobi-like forms and quasimodular polynomials with character.

10. Shimura Correspondences

In the classical theory of modular forms, a Shimura correspondence provides a map from from half integral weight modular forms to integral weight modular forms that is Hecke equivariant. In this section we consider similar maps for quasimodular forms.

First, we review Shimura's construction of a Hecke-equivariant map from half integral weight cusp forms to integral weight cusp forms (see [23]). Let χ be a Dirichlet character, so that the associated L-function is given by

$$L(s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

Given a half integer λ with 2λ odd, let $S_\lambda(\Gamma_0(N), \chi)$ be the space of cusp forms for $\Gamma_0(N)$ of weight λ with character χ. We consider an element $f \in S_\lambda(\Gamma_0(N), \chi)$, so that it satisfies

$$(f \mid_\lambda \gamma)(z) = \widehat{\mathfrak{J}}(\gamma, z)^{-2\lambda} f(\gamma z) = \chi(d) f(z)$$

for all $\gamma \in \Gamma_0(N)$ and $z \in \mathcal{H}$, where d is the $(2, 2)$-entry of γ. We recall that the Hecke operators $T_{\lambda, \chi}^N(\tilde{\alpha})$ on $S_\lambda(\Gamma_0(4N), \chi)$ are given by

$$T_{\lambda, \chi}^N(\tilde{\alpha})f = (\det \alpha)^{-\lambda-1} \sum_{\nu=1}^{r} \chi(\alpha_\nu)(f \mid_\lambda \alpha_\nu).$$

The following is the main result obtained by Shimura in [23].

Theorem 10.1. *Suppose that $g(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_{k+1/2}(\Gamma_0(4N), \chi)$ with $q = e^{2\pi i z}$ is a half-integral weight cusp form with $k \geq 1$. Let t be a positive square-free integer, and define the Dirichlet character ψ_t by $\psi_t(n) = \chi(n)(\frac{-1}{n})^k(\frac{t}{n})$. If the complex numbers $A_t(n)$ are defined by*

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = L(s - k + 1, \psi_t) \sum_{n=1}^{\infty} \frac{b(tn^2)}{n^s},$$

then the function

$$S_{t,k}(g(z)) = \sum_{n=1}^{\infty} A_t(n) q^n$$

is a weight $2k$ modular form belonging to $M_{2k}(\Gamma_0(2N), \chi^2)$. If $k \geq 2$ then $S_{t,k}(g(z))$ is a cusp form. Furthermore, if $k = 1$, then $S_{t,1}(g(z))$ is a cusp form if $g(z)$ is in the orthogonal complement of the subspace of $S_{3/2}(\Gamma_0(4N), \chi)$ spanned by single variable theta functions.

From Theorem 10.1 we obtain the Hecke equivariant Shimura map

$$\mathrm{Sh}_{\lambda,t,\chi} : S_\lambda(\Gamma_0(4N), \chi) \to M_{2\lambda-1}(\Gamma_0(2N), \chi^2)$$

defined by

$$\mathrm{Sh}_{\lambda,t,\chi}(f(z)) = \sum_{n=1}^{\infty} A_{\lambda,t}(n) q^n$$

for

$$f(z) = \sum_{n=1}^{\infty} a_\lambda(f, n) q^n,$$

where

$$A_{\lambda,t}(n) = L(s - \lambda + 5/2, \chi_t) a_\lambda(f, n^2 t)$$

for all $n \geq 1$.

In order to discuss the quasimodular analog of the Shimura map, we consider a quasimodular polynomial $F(z, X) \in QP^{m-r}_{\lambda+2m}(\Gamma_0(4N), \chi)$ of the form

$$(10.1) \qquad F(z, X) = \sum_{u=0}^{m-r} f_u(z) X^u$$

for $z \in \mathcal{H}$, and set

$$(10.2) \qquad (_Q\mathrm{Sh}^{m,m',r}_{\lambda,t,\chi} F)(z, X) = \sum_{\ell=0}^{m'-r} \frac{\Gamma(m' - r + 1)\Gamma(2\lambda + 4r) \, \mathrm{Sh}^{(m'-r-\ell)}_{\lambda+2r,t,\chi}(f_{m-r})(z)}{\Gamma(\ell + 1)\Gamma(m' - r - \ell + 1)\Gamma(m' - \ell + 2\lambda + 3r)} X^\ell,$$

which is an element of $\mathcal{F}_{m'-r}[X]$ with $m' \geq 0$.

Proposition 10.2. *The formula (10.2) determines the Hecke equivariant linear map*

$$_Q\mathrm{Sh}^{m,m',r}_{\lambda,t,\chi} : QP^{m-r}_{\lambda+2m}(\Gamma_0(4N), \chi) \to QP^{m'-r}_{2(\lambda+m'+r)-1}(\Gamma_0(2N), \chi^2)$$

for $m, m' \geq 0$ and $r \leq \min\{m, m'\}$.

Proof. We first note that that the map $\mathfrak{S}_m$ in (3.8) can be extended to the half integral case, so that

$$\mathfrak{S}_m(QP^m_\lambda(\Gamma, \chi)) \subset M_{\lambda-2m}(\Gamma, \chi).$$

The lifting formula (5.2) can also be modified so that for each $h \in M_\mu(\Gamma, \chi)$ with $\mu \in \frac{1}{2}\mathbb{Z}$ the polynomial

$$(10.3) \qquad (\Xi^{2\mu}_m h)(z, X) = \sum_{\ell=0}^{m} \frac{\Gamma(m + 1)\Gamma(2\mu) h^{(m-\ell)}(z)}{\Gamma(\ell + 1)\Gamma(m - \ell + 1)\Gamma(m - \ell + 2\mu)} X^\ell$$

is a quasimodular polynomial belonging to $QP^m_{\mu+2m}(\Gamma, \chi)$ and satisfies

$$\mathfrak{S}_m(\Xi^{2\mu}_m h) = h.$$

We now consider a quasimodular polynomial $F(z, X) \in QP_{\lambda+2m}^{m-r}(\Gamma_0(4N), \chi)$ given by (10.1), so that

$$\mathfrak{S}_{m-r}(F) \in M_{\lambda+2r}(\Gamma_0(4N), \chi),$$
$$\mathrm{Sh}_{\lambda+r,t,\chi}(\mathfrak{S}_{m-r}F) \in M_{2\lambda+4r-1}(\Gamma_0(2N), \chi^2).$$

Thus, using (10.2) and (10.3), we see that

$$(10.4) \qquad \Xi_{m'-r}^{2\lambda+4r-1}(\mathrm{Sh}_{\lambda+r,t,\chi}(\mathfrak{S}_{m-r}F))$$

$$= \sum_{\ell=0}^{m'-r} \frac{\Gamma(m'-r+1)\Gamma(2\lambda+4r)\,\mathrm{Sh}_{\lambda+2r,t,\chi}^{(m'-r-\ell)}(f_{m-r})}{\Gamma(\ell+1)\Gamma(m'-r-\ell+1)\Gamma(m'-\ell+2\lambda+3r)} X^\ell$$

$$= {}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',r}(F);$$

hence it follows that

$${}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',r}(F) \in QP_{2(\lambda+m'+r)-1}^{m'-r}(\Gamma_0(2N), \chi^2).$$

On the other hand, the maps $\mathfrak{S}_{m-r}$ and $\Xi_{m'-r}^{2\lambda+4r-1}$ are Hecke equivariant by the commutativity of the diagrams (4.12) and (5.4), respectively; hence the Hecke equivariance of ${}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',r}$ follows from the same property for the Shimura map $\mathrm{Sh}_{\lambda+r,t,\chi}$ of modular forms. $\square$

From (10.4) we obtain the relation

$${}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',r} = \Xi_{m'-r}^{2\lambda+4r-1} \circ \mathrm{Sh}_{\lambda+r,t,\chi} \circ \mathfrak{S}_{m-r}$$

and therefore the Hecke equivariant commutative diagram

$$
\begin{array}{ccc}
QP_{\lambda+2m}^{m-r}(\Gamma_0(4N), \chi) & \xrightarrow{{}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',r}} & QP_{2(\lambda+r+m')-1}^{m'-r}(\Gamma_0(2N), \chi') \\
{\scriptstyle \mathfrak{S}_{m-r}} \downarrow & & \uparrow {\scriptstyle \Xi_{m'-r}^{2\lambda+4r-1}} \\
S_{\lambda+2r}(\Gamma_0(4N), \chi) & \xrightarrow{\mathrm{Sh}_{\lambda+r,\chi}} & S_{2(\lambda+2r)-1}(\Gamma_0(2N), \chi^2)
\end{array}
$$

for each $r \in \{0, \dots, m\}$.

Assuming that $m' \geq m$, we now consider a quasimodular polynomial

$$G(z, X) \in QP_{\lambda+2m}^m(\Gamma_0(4N), \chi),$$

and define for each $r \in \{0, 1, \dots, m\}$ the quasimodular polynomial

$$G_r(z, X) \in QP_{\lambda+2m}^{m-r}(\Gamma_0(4N), \chi)$$

by setting

$$G_0 = G, \quad G_{\ell+1} = G_\ell - (m-\ell)!(\Pi_{m-\ell} \circ \widehat{\Xi}_0 \circ \mathfrak{S}_{m-\ell})G_\ell \in QP_{\lambda+2m}^{m-\ell}(\Gamma_0(4N), \chi)$$

for $0 \leq \ell \leq m-1$, where $\widehat{\Xi}_0$ is as in (5.5). Then, if we set

$$\bigoplus_{r=0}^m {}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',r}(G) = ({}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',0} G_0, {}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',1} G_1, \dots, {}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',m} G_m)$$

we obtain the complex linear map

$$\bigoplus_{r=0}^m {}_Q\mathrm{Sh}_{\lambda,t,\chi}^{m,m',r} : QP_{\lambda+2m}^m(\Gamma_0(4N), \chi) \to \bigoplus_{r=0}^m QP_{2(\lambda+r+m')-1}^{m'-r}(\Gamma_0(2N), \chi^2),$$

which is Hecke equivariant.

11. Shintani Liftings

It was Shintani (cf. [24]) who constructed Hecke equivariant maps from integral weight λ cusp forms to half-integral weight cusp forms, which may be regarded as inverses of Shimura maps. In this section we study the quasimodular version of Shintani maps.

We first review the construction of Shintani maps for modular forms. Let $\mathfrak{Q}$ be the space of integral indefinite binary quadratic forms of the form

$$Q = Q(X, Y) = aX^2 + bXY + cY^2 = [a, b, c]$$

with $\mathrm{Disc}(Q) = b^2 - 4ac > 0$. Given a positive integer M, we set

$$\mathfrak{Q}_M = \{Q(X, Y) = [a, b, c] \in \mathfrak{Q} \mid (a, M) = 1, \ b \equiv c \equiv 0 \ (\mathrm{mod} \ M)\}$$

if M is odd, and

$$\mathfrak{Q}_M = \{Q(X, Y) = [a, b, c] \in \mathfrak{Q} \mid (a, M) = 1, \ b \equiv 0 \ (\mathrm{mod} \ 2M), \ c \equiv 0 \ (\mathrm{mod} \ M)\}$$

if M is even. Then the congruence subgroup $\Gamma_0(M)$ acts on $\mathfrak{Q}_M$ on the left by

$$(\gamma \cdot Q)(X, Y) = Q((X, Y)\gamma^{-1})$$

for all $\gamma \in \Gamma_0(M)$ and $Q \in \mathfrak{Q}$. Following [24], to each integral indefinite binary form $Q \in \mathfrak{Q}_M$ we associate a pair of points $\omega_Q, \omega'_Q \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{i\infty\}$ given by

$$(\omega_Q, \omega'_Q) = \begin{cases} \left(\frac{b+\sqrt{\mathrm{Disc}(Q)}}{2c}, \frac{b-\sqrt{\mathrm{Disc}(Q)}}{2c}\right) & \text{if } c \neq 0; \\ (i\infty, \frac{a}{b}) & c = 0 \text{ and } b > 0; \\ (\frac{a}{b}, i\infty) & c = 0 \text{ and } b < 0. \end{cases}$$

Given $Q \in \mathfrak{Q}$, we denote by γ_Q the unique generator of the stabilizer of $Q \in \mathfrak{Q}_M$ in the congruence group $\Gamma_0(M)$. We then consider the path C_Q in $\mathcal{H}$ defined by

$$C_Q = \begin{cases} \overline{(\omega_Q, \omega'_Q)} & \text{if } \mathrm{Disc}(Q) \text{ is a perfect square}; \\ \overline{(\omega, \gamma_Q \omega)} & \text{otherwise}, \end{cases}$$

where ω is an arbitrary point in $\mathbb{P}^1(\mathbb{Q})$ and $\overline{(\cdot, \cdot)}$ denotes the oriented geodesic path joining the given pair of points.

We write

$$\chi(Q) = \chi(a)$$

if $Q = [a, b, c] \in \mathfrak{Q}_M$. Given $f \in S_{2\lambda-1}(\Gamma_0(M), \chi^2)$, we also set

$$\Theta_{\lambda, \chi}(f) = \begin{cases} \sum_{Q \in \mathfrak{Q}/\Gamma_0(M)} I_{\lambda, \chi} q^{\frac{\mathrm{Disc}(Q)}{M}} & \text{if } M \text{ is odd}; \\ \sum_{Q \in \mathfrak{Q}/\Gamma_0(M)} I_{\lambda, \chi} q^{\frac{\mathrm{Disc}(Q)}{4M}} & \text{if } M \text{ is even}, \end{cases}$$

and

$$I_{\lambda, \chi}(f, Q) = \chi(Q) \int_{C_Q} f(\tau) Q(1, -\tau)^{\lambda - \frac{3}{2}} d\tau.$$

We denote by $S_\lambda(\Gamma_0(4M), \chi')$ the space of cups forms of level $4M$, half integral weight λ, and Nebentype character χ'. The following result was obtained by Shintani.

If χ is a Dirichlet character χ defined modulo M, we define the associated Nebentype character χ' modulo M by

$$\chi'(d) = \chi(d) \left(\frac{(-1)^{2\lambda+1} M}{d}\right)$$

for all $d \in (\mathbb{Z}/4M\mathbb{Z})^\times$.

Theorem 11.1. *Let χ be a Dirichlet character defined modulo M. Then for each $f \in S_{2\lambda-1}(\Gamma_0(M), \chi^2)$, the series $\Theta_{\lambda,\chi}(f)$ is the q-expansion of a half-integral weight cusp form in $S_\lambda(\Gamma_0(4M), \chi')$. Moreover, the map*

$$\Theta_{\lambda,\chi} : S_{2\lambda-1}(\Gamma_0(M), \chi^2) \to S_\lambda(\Gamma_0(4M), \chi')$$

is a Hecke-equivalent $\mathbb{C}$-linear map.

Proof. See [24]. $\square$

To introduce a quasimodular version of Shintani's result, we consider a quasimodular polynomial $F(z, X) \in QP^{m-r}_{2\lambda-1+2m}(\Gamma_0(4M), \chi^2)$ of the form

$$F(z, X) = \sum_{u=0}^{m-r} f_u(z) X^u$$

for $z \in \mathcal{H}$, and set

(11.1) $\qquad (_Q\Theta^{m,m',r}_{\lambda,\chi} F)(z, X) = \sum_{\ell=0}^{m'-r} \frac{\mathbf{\Gamma}(m' - r + 1)\mathbf{\Gamma}(2\lambda + +2r)\Theta^{(m'-r-\ell)}_{\lambda+r,\chi}(f_{m-r})(z)}{\mathbf{\Gamma}(\ell + 1)\mathbf{\Gamma}(m' - r - \ell + 1)\mathbf{\Gamma}(m' - \ell + 2\lambda + r)} X^\ell,$

which is an element of $\mathcal{F}_{m'-r}[X]$ with $m' \geq 0$.

Proposition 11.2. *The formula (11.1) determines the Hecke equivariant linear map*

$$_Q\Theta^{m,m',r}_{\lambda,\chi} : QP^{m-r}_{2\lambda-1+2m}(\Gamma_0(M), \chi^2) \to QP^{m'-r}_{2m'+\lambda-r}(\Gamma_0(4M), \chi')$$

for $m, m' \geq$ and $0 \leq r \leq \min\{m, m'\}$.

Proof. Given a quasimodular polynomial $F(z, X) \in QP^{m-r}_{2\lambda-1+2m}(\Gamma_0(M), \chi^2)$, we have

$$\mathfrak{S}_{m-r} F \in S_{2(\lambda+r)-1}(\Gamma_0(M), \chi^2),$$

$$\Theta_{\lambda+r,\chi}(\mathfrak{S}_{m-r} F) \in S_{\lambda+r}(\Gamma_0(4M), \chi').$$

Thus, using (10.3)and (11.1), we see that

(11.2) $\qquad \Xi^{2(\lambda+r)}_{m'-r}(\Theta_{\lambda+r,\chi}(\mathfrak{S}_{m-r} F))$

$$= \sum_{\ell=0}^{m'-r} \frac{\mathbf{\Gamma}(m' - r + 1)\mathbf{\Gamma}(2\lambda + 2r)\Theta^{(m'-r-\ell)}_{\lambda+r,\chi}(f_{m-r})(z)}{\mathbf{\Gamma}(\ell + 1)\mathbf{\Gamma}(m' - r - \ell + 1)\mathbf{\Gamma}(m' - \ell + 2\lambda + r)} X^\ell$$

$$= {}_Q\Theta^{m,m',r}_{\lambda,\chi}(F);$$

hence it follows that

$$_Q\Theta^{m,m',r}_{\lambda,\chi}(F) \in QP^{m'-r}_{2m'+\lambda-r}(\Gamma_0(4M), \chi').$$

On the other hand, the maps $\mathfrak{S}_{m-r}$ and $\Xi^{2(\lambda+r)}_{m'-r}$ are Hecke equivariant by the commutativity of the diagrams (4.12) and (5.4), respectively; hence the Hecke equivariance of $_Q\Theta^{m,m',r}_{\lambda,\chi}$ follows from the same property for the Shintani map $\Theta_{\lambda+r,\chi}$ of modular forms. $\square$

From (11.2) we obtain the relation

$$_Q\Theta_{\lambda,\chi}^{m,m',r} = \Xi_{m'-r}^{2(\lambda+r)} \circ \Theta_{\lambda+r,\chi} \circ \mathfrak{S}_{m-r}$$

as well as the Hecke equivariant commutative diagram

$$\begin{array}{ccc}
QP_{2\lambda-1+2m}^{m-r}(\Gamma_0(M),\chi^2) & \xrightarrow{_Q\Theta_{\lambda,\chi}^{m,m',r}:} & QP_{2m'+\lambda-r}^{m'-r}(\Gamma_0(4M),\chi') \\
\mathfrak{S}_{m-r}\downarrow & & \uparrow\Xi_{m'-r}^{2(\lambda+r)} \\
S_{2(\lambda+r)-1}(\Gamma_0(M),\chi^2) & \xrightarrow{\Theta_{\lambda+r,\chi}} & S_{\lambda+r}(\Gamma_0(4M),\chi')
\end{array}$$

for each $r \in \{0,1,\ldots,m\}$.

We now consider a quasimodular polynomial $G(z,X) \in QP_{2\lambda-1+2m}^{m-r}(\Gamma_0(M),\chi^2)$, and define the associated quasimodular polynomials $G_r(z,X) \in QP_{2\lambda-1+2m}^{m-r}(\Gamma_0(M),\chi^2)$ with $m' \geq m$ for each $0 \leq r \leq m$ by

$$G_0 = G, \quad G_{\ell+1} = G_\ell - (m-\ell)!(\Pi_{m-\ell} \circ \widehat{\widehat{\Xi}}_0 \circ \mathfrak{S}_{m-\ell})G_\ell \in QP_{2\lambda-1+2m}^{m-\ell}$$

for $0 \leq \ell \leq m-1$. Then by setting

$$\bigoplus_{r=0}^{m} {}_Q\Theta_{\lambda,\chi}^{m,m',r}(G) = ({}_Q\Theta_{\lambda,\chi}^{m,m',0}G_0, {}_Q\Theta_{\lambda,\chi}^{m,m',1}G_1, \ldots, {}_Q\Theta_{\lambda,\chi}^{m,m',m}G_m)$$

we obtain the $\mathbb{C}$-linear map

$$\bigoplus_{r=0}^{m} {}_Q\Theta_{\lambda,\chi}^{m,m',r} : QP_{2\lambda-1+2m}^{m}(\Gamma_0(M),\chi^2) \to \bigoplus_{r=0}^{m} QP_{2m'+\lambda-r}^{m'-r}(\Gamma_0(4M),\chi'),$$

which is Hecke equivariant.

References

[1] N. O. Azaiez, *The ring of quasimodular forms for a cocompact group*, J.Number Theory **128** (2008), 1966-1988.

[2] V. Bosser and F. Pellarin, *On certain families of Drinfeld quasi-modular forms*, J. Number Theory, **129** (2009), 2952-2990.

[3] J. H. Bruinier, G. Van Der Geer, G. Harder and D. Zagier, The 1, 2, 3 of modular forms, Lectures at a Summer school in Nordfjordeid, Norway, Springer, Heidelberg, 2007.

[4] Y. Choie, *Jacobi forms and the heat operator*, Math. Z. **225** (1997), 95–101.

[5] Y. Choie, *Jacobi forms and the heat operator*, II. Illinois J. Math. **42** (1998), 179–186.

[6] Y. Choie and M. H. Lee, *Rankin-Cohen brackets on pseudodifferential operators*, J. Math. Anal. Appl. **326** (2007), 882–895.

[7] P. B. Cohen, Y. Manin, and D. Zagier, *Automorphic pseudodifferential operators*, Algebraic Aspects of Nonlinear Systems, Birkhäuser, Boston, 1997, pp. 17–47.

[8] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math., vol. 55, Birkhäuser, Boston, 1985.

[9] A. Eskin and A. Okounkov, *Pillowcases and quasimodular forms*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser, Boston, 2006, pp. 1–25.

[10] A. Eskin and A. Okounkov, *Asymptotics of numbers of branched coverings of a torus and volumes of moduli spaces of holomorphic differentials*, Invent. Math. **145** (2001), 59–103.

[11] M. Kaneko and M. Koike, *On extremal quasimodular forms*, Kyushu J. Math. **60** (2006), 457–470.

[12] M. Kaneko, *On modular forms of weight $(6n+1)/5$ satisfying a certain differential equation*, Number theory, Dev. Math., vol. 15, Springer, New York, 2006, pp. 97–102.

[13] M. Kaneko and Zagier, *A generalized Jacobi theta function and quasimodular forms*, Progress in Math., vol. 129, Birkhäuser, Boston, 1995, pp. 165–172.

[14] M. H. Lee, *Lie algebras of formal power series*, Rev. Mat. Complut. **20** (2007), 463–381.

[15] M. H. Lee, *Radial heat operators on Jacobi-like forms*, Math. J. Okayama Univ. **51** (2009), 27–46.

[16] M. H. Lee and H. C. Myung, *Hecke operators on Jacobi-like forms*, Canad. Math. Bull. **44** (2001), 282–291.

[17] S. Leliévre and E. Royer,*Orbitwise countings in $H(2)$ and quasimodular forms*, Int. Math. Res. Not. 2006, Art. ID 42151, 30 pp

[18] F. Martin and E. Royer, *Rankin-Cohen brackets on quasimodular forms*, math.NT/050965.

[19] T. Miyake, *Modular forms*, Springer-Verlag, Heidelberg, 1989.

[20] H. Movasati, *On differential modular forms and some analytic relations between Eisenstein series*, Ramanujan J. **17** (2008), 53–76.

[21] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory, and completed cycles*, Ann. of Math. **163** (2006), 517–560.

[22] E. Royer, *Evaluating convolution sums of the divisor function via quasimodular forms*, Int. J. Number Theory **21** (2007), 231–262.

[23] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. **97** (1973), 440–481.

[24] G. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. **58** (1975), 83–126.

[25] D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), 57–75.

DEPARTMENT OF MATHEMATICS AND PMI(POHANG MATHEMATICAL INSTITUTE), POSTECH, POHANG 790-784, KOREA

E-mail address: yjc@postech.ac.kr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTHERN IOWA, CEDAR FALLS, IA 50614, U.S.A.

E-mail address: lee@math.uni.edu